Derivation of the Reduction of Order Formula for a Second Solution in Section 4.2

Suppose we have the general homogeneous linear 2nd-order differential equation
\[ y'' + P(x) \cdot y' + Q(x) \cdot y = 0 \]  
where \( P \) and \( Q \) are continuous on some interval \( I \), and suppose \( y_1(x) \) is a solution to the DE on \( I \) such that \( y_1(x) \neq 0 \) for all \( x \in I \).

We seek a second solution \( y_2 \) such that \( y_1 \) and \( y_2 \) are linearly independent. Because the ratio of two linearly independent functions is non-constant, we let \( y = u(x) \cdot y_1(x) \) for some nonconstant function \( u \). We will assume that \( y \) is a solution and attempt to find \( u \). Hence,
\[ y' = u'y_1 + u'y_1 \]
and
\[ y'' = u'y_1'' + u'y_1' + u'y_1' + u''y_1 \]
\[ = u'y_1'' + 2u'y_1' + u''y_1. \]

Thus, the DE \( y'' + Py' + Qy = 0 \) gives
\[ u'y_1'' + 2u'y_1' + u''y_1 + P(u'y_1' + u'y_1) + Q(u'y_1) = 0 \]
\[ u \left( y_1'' + Py_1' + Qy_1 \right) + u'(2y_1' + Py_1) + u''y_1 = 0 \]
\[ \text{zero (} y_1 \text{ is a solution)} \]
\[ u'(2y_1' + Py_1) + u''y_1 = 0. \]

Let \( w = u' \) (to reduce the order of the DE). The equation becomes
\[ w(2y_1' + Py_1) + w'y_1 = 0, \]
which is linear and separable. Writing the equation in standard linear form gives
\[ w' + \left( \frac{2y_1' + Py_1}{y_1} \right)w = 0 \]
\[ w' + \left( 2\frac{y_1'}{y_1} + P \right)w = 0. \]

Hence, our integrating factor will be \( I = e^{\int \left( \frac{2y_1'}{y_1} + P \right) dx} = e^{\ln|y_1|^2 + \int P \, dx + C_0} = C_1y_1^2e^{\int P \, dx}. \)
Multiplying our equation by \( I \), integrating, and solving for \( w \) gives...
\[
\frac{d}{dx}\left[ \left( C_1 y_1^2 e^{\int Pdx} \right) \cdot w \right] = 0
\]
\[
C_1 y_1^2 e^{\int Pdx} \cdot w = C_2
\]
\[
w = \frac{C_2}{C_1 y_1^2 e^{\int Pdx}}
\]
\[
w = C_3 e^{-\int Pdx} y_1^2.
\]
Because \( u' = w \), we have
\[
u' = C_3 e^{-\int Pdx} y_1^2
\]
\[
u = \int C_3 e^{-\int Pdx} y_1^2 \, dx + C_4.
\]
Thus, \( y = u \cdot y_1 \) implies that
\[
y = y_1 \cdot C_3 e^{\int P(x)dx} y_1^2 \, dx + C_4.
\]
By choosing \( C_3 = 1 \) and \( C_4 = 0 \), we obtain
\[
y_2 = y_1 e^{\int P(x)dx} y_1^2 \, dx.
\]
The Wronskian for \( y_1 \) and \( y_2 \) is given by
\[
W(y_1, y_2) = \begin{vmatrix}
y_1 & y_1 e^{\int P(x)dx} y_1^2 \, dx \\
y_1' e^{-\int P(x)dx} y_1^2 + y_1 e^{\int P(x)dx} y_1^2 \, dx
\end{vmatrix}
\]
\[
= y_1 \cdot \left( e^{\int P(x)dx} y_1 + y_1' e^{\int P(x)dx} y_1^2 \, dx \right) - \left( y_1 e^{\int P(x)dx} y_1^2 \, dx \right) \cdot y_1'
\]
\[
= e^{\int P(x)dx} y_1 + y_1 e^{\int P(x)dx} y_1^2 \, dx - y_1 e^{\int P(x)dx} y_1^2 \, dx
\]
\[
= e^{\int P(x)dx} \neq 0 \text{ for all } x \in I,
\]
hence \( y_1 \) and \( y_2 \) are linearly independent.

Thus, if \( y_1 \) is a solution of a homogeneous linear 2nd-order differential equation, then a second
solution $y_2$ for which $y_1$ and $y_2$ are linearly independent is

$$y_2 = y_1 e^{-\int \frac{p(x)dx}{y_1^2}} dx.$$