Setting Up Partial Fraction Decompositions

Let \( P(x) \) be the numerator of a proper rational expression we wish to decompose, and suppose the denominator is in factored form. The types of factors in the denominator and the power to which each factor is raised determine the number of partial fractions that are obtained and the form of the numerator of each partial fraction.

**Linear Factors**

If the factors of the denominator are all linear factors raised only to the first power, then each denominator factor determines one partial fraction, each of which has a constant numerator:

\[
\frac{P(x)}{(a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)} = \frac{A}{a_1x+b_1} + \frac{B}{a_2x+b_2} + \cdots + \frac{M}{a_nx+b_n}
\]

Note that capital letters \((A, B, C, \text{etc.})\) are used to represent unknown numerical (constant) values in the numerators.

If any of the factors of the denominator are linear factors raised to some power \(n\) (other than one), we call those factors **repeated linear factors**. A linear factor raised to the \(n\)th power produces \(n\) distinct partial fractions with denominators that are raised to successive powers of that linear factor from 1 to \(n\):

\[
\frac{P(x)}{(cx+d)^n} = \frac{A}{cx+d} + \frac{B}{(cx+d)^2} + \cdots + \frac{N}{(cx+d)^n}
\]

It is not uncommon to have a denominator that has both types of denominator factors already described. The example below shows how to set up such a decomposition.

\[
\frac{P(x)}{x(3x-1)(2x+5)} = \frac{A}{x} + \frac{B}{3x-1} + \frac{C}{2x+5} + \frac{D}{(2x+5)^2}
\]

**Irreducible Quadratic Factors**

Quadratic factors of the denominator that do not factor (for example, \(x^2 + 1\) or \(3x^2 + x + 5\)) are called **irreducible quadratic factors**. Each irreducible quadratic factor in the denominator determines one partial fraction in the decomposition, each of which has a numerator in linear form:

\[
\frac{P(x)}{(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)} = \frac{Ax+B}{a_1x^2+b_1x+c_1} + \frac{Cx+D}{a_2x^2+b_2x+c_2}
\]

Similar to repeated linear factors, a denominator may contain repeated quadratic factors:

\[
\frac{P(x)}{(ax^2+bx+c)^n} = \frac{Ax+B}{ax^2+bx+c} + \frac{Cx+D}{(ax^2+bx+c)^2} + \frac{Ex+F}{(ax^2+bx+c)^3}
\]

A particular rational expression to be decomposed may have both types of factors (linear and irreducible quadratic). Here is an example:

\[
\frac{2x+5}{x^2(x-1)(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+4}
\]

Note that \(x^2\) is NOT an irreducible quadratic factor—it is a repeated linear factor (think \(x^2 = x \cdot x\)).