TRANFORMATIONS OF FUNCTIONS

For the purposes of this course, there are eight basic transformations of functions, each of which falls into one of two categories: Vertical Transformations, and Horizontal Transformations. A brief summary of the two categories is below.

<table>
<thead>
<tr>
<th>Vertical Transformations</th>
<th>Horizontal Transformations</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Related to a change in y-values</td>
<td>- Related to a change in x-values</td>
</tr>
<tr>
<td>- Affects the range, but not domain</td>
<td>- Affects the domain, and sometimes range</td>
</tr>
<tr>
<td>- Up-Down changes</td>
<td>- Left-Right changes</td>
</tr>
<tr>
<td>- (Typically) Intuitive</td>
<td>- (Typically) counterintuitive</td>
</tr>
</tbody>
</table>

1. Vertical Shift: \( y = f(x) + k \)
   [Generally: \( y – k = f(x) \) ]

2. Vertical Stretch/Compression:
   \( y = k f(x) \) (k constant)

3. Reflection about the X-Axis:
   \( y = –f(x) \)

4. Absolute Value: \( y = |f(x)| \)

5. Horizontal Shift: \( y = f(x + k) \)

6. Horizontal Stretch/Compression
   \( y = f(ax) \) (k constant)

7. Reflection about the Y-Axis
   \( y = f(–x) \)

8. Absolute Value: \( y = f(\ |x| \) )
   Often affects range.

For a more detailed discussion of the transformations the following graph will be used (from: Algebra & Trigonometry 6th Ed., p. 276, Chapter Review Exercises, #6):

\[
y = f(x) = \begin{cases} 
  1 & \text{if } -5 \leq x \leq -1 \\
  -x & \text{if } -1 < x < 3 \\
  3x - 12, & \text{if } 3 \leq x \leq 4 
\end{cases}
\]

**Properties:**

(I) Domain: \( x \in [-5, 4] \)

(II) Range: \( y \in [-3, 1] \).

(III) y-intercept: (0,0); x-intercepts: (0,0), (4,0)

(IV) Symmetry: None

For the purposes of this paper, \( \text{Dom}_f \) represents the domain of \( f(x) \) and \( \text{Range}_f \) represents the range of \( f(x) \).

On the graphs, I represent the graphs of \( f(x) \) with a dotted line: 
I represent the graphs of the transformations with a solid line:
1. Vertical Shift

\[ y = f(x) + k, \quad k \text{ constant} \]

**General Properties:**

(I) **Domain:** \( \text{Dom}_f \) (Domain is unaffected in vertical transformation).

(II) **Range:** \( k + \text{Range}_f \) (Add \( k \) to each value in the Range of \( f(x) \)).

(III) **Symmetry:** Changes origin symmetry but preserves y-axis symmetry (that is, even functions remain even; odd functions may no longer be odd functions).

(IV) If \( k > 0 \), then graph moves \text{up} \( |k| \) places. If \( k < 0 \), then graph moves \text{down} \( |k| \) places.

**Example:**

\[ y = f(x) + 3 \]

\[ (-5,4) \quad (-1,4) \quad (0,3) \quad (4,3) \]

\[ (-5,1) \quad (-1,1) \quad (0,0) \quad (3,0) \quad (0,0) \]

\[ (3,-3) \quad (4,0) \quad (3,-3) \]

**Domain:** \( x \in [-5, 4] \)

**Range:** \( y \in [0, 4] = [-3+3, 1+3] \).

\[ y = f(x)+3 = \begin{cases} 
1+3 & \text{if } -5 \leq x \leq -1 \\
-x +3 & \text{if } -1 < x < 3 \\
3x -12 +3, & \text{if } 3 \leq x \leq 4 
\end{cases} = \begin{cases} 
4 & \text{if } -5 \leq x \leq -1 \\
-x +3 & \text{if } -1 < x < 3 \\
3x -9, & \text{if } 3 \leq x \leq 4 
\end{cases} 
\]

**Note:** Add 3 to every \( f(x) \) value.

2. Horizontal Shift

\[ y = f(x + k), \quad k \text{ constant} \]

**General Properties:**

(I) **Domain:** \( \text{Dom}_f - k \) (Subtract \( k \) from each value in the domain of \( f(x) \)).

(II) **Range:** \( \text{Range}_f \) (Range of \( f(x) \) unaffected in this horizontal transformation).

(III) **Symmetry:** Destroys both origin symmetry and y-axis symmetry (typically).

(IV) If \( k > 0 \), then graph moves \text{left} \( |k| \) places. If \( k < 0 \), then graph moves \text{right} \( |k| \) places.

**Example:**

\[ y = f(x - 2) \]

\[ (-5,1) \quad (-3,1) \quad (-1,1) \quad (1,1) \quad (2,0) \quad (4,0) \quad (6,0) \]

\[ (0,0) \quad (2,0) \quad (3,-3) \quad (5,-3) \]

**Domain:** \( x \in [-3, 6] = [-5+2, 4+2] \)

**Range:** \( y \in [-3, 1] \).

\[ y = f(x-2) = \begin{cases} 
1, & \text{if } -5 \leq x - 2 \leq -1 \\
-(x-2), & \text{if } -1 < x - 2 < 3 \\
3(x-2) -12, & \text{if } 3 \leq x - 2 \leq 4 
\end{cases} = \begin{cases} 
1 & \text{if } -3 \leq x \leq 1 \\
-x +2 & \text{if } 1 < x < 5 \\
3x -18, & \text{if } 5 \leq x \leq 6 
\end{cases} 
\]

**Note:** Replace EACH \( x \) with \( (x - 2) \), including in the domain statement then simplify.
3. **Vertical Stretch/Compression**

\[ y = k \cdot f(x), \quad k > 0 \text{ a constant} \]

**General Properties:**

(I) **Domain:** \( \text{Dom}_f \) (Domain is unaffected in vertical transformation)

(II) **Range:** \( k \times \text{Range}_f \) (Multiply each value in the Range of \( f(x) \) by \( k \)).

(III) **Symmetry:** Preserves both origin symmetry and y-axis symmetry.

(IV) If \( 0 < k < 1 \), then graph **compresses** (shrinks). If \( k > 1 \), then graph **stretches**. Note that values on the x-axis (\( y = 0 \)) do not change.

**Example:** \( y = 2f(x) \)

\[
\begin{align*}
\text{Domain: } & x \in [-5, 4] \\
\text{Range: } & y \in [-6, 2] = [-3 \times 2, 1 \times 2].
\end{align*}
\]

\[
y = 2f(x) = \begin{cases} 
2(1) & \text{if } -5 \leq x \leq -1 \\
2(-x) & \text{if } -1 < x < 3 \\
2(3x - 12), & \text{if } 3 \leq x \leq 4
\end{cases}
\]

**Note:** Multiply each \( f(x) \) value by 2.

4. **Horizontal Stretch/Compression**

\[ y = f(kx), \quad k > 0 \text{ a constant} \]

**General Properties:**

(I) **Domain:** \( \text{Dom}_f \div k \) (Divide each value in the domain of \( f(x) \) by \( k \) (or multiply by \( 1/k \))).

(II) **Range:** \( \text{Range}_f \) (Range of \( f(x) \) unaffected in this horizontal transformation).

(III) **Symmetry:** Preserves both origin symmetry and y-axis symmetry.

(IV) If \( 0 < k < 1 \), then graph **stretches**. If \( k > 1 \), then graph **compresses** (shrinks). Note that value on the y-axis (\( x = 0 \)) does not change.

**Example:** \( y = f(2x) \)

\[
\begin{align*}
\text{Domain: } & x \in [-2.5, 2] = [-5 \div 2, 4 \div 2] \\
\text{Range: } & y \in [-3, 1].
\end{align*}
\]

\[
y = f(2x) = \begin{cases} 
1 & \text{if } -5 \leq 2x \leq -1 \\
-(2x) & \text{if } -1 < 2x < 3 \\
3(2x) - 12, & \text{if } 3 \leq 2x \leq 4
\end{cases}
\]

\[
= \begin{cases} 
1 & \text{if } -2.5 \leq x \leq -0.5 \\
-2x & \text{if } -0.5 < x < 1.5 \\
6x - 12, & \text{if } 1.5 \leq x \leq 2
\end{cases}
\]

**Note:** Replace EACH \( x \) with \( 2x \), including in the domain statement then simplify.
5. Reflection About the X-Axis
\[ y = -f(x) \]

**General Properties:**
(I) **Domain:** \( \text{Dom}_f \) (Domain is unaffected in vertical transformation)
(II) **Range:** \(-1 \times \text{Range}_f\) (Interchange the signs of the endpoints of the Range of \(f(x)\)).
(III) **Symmetry:** Preserves both origin symmetry and y-axis symmetry.
(IV) All positive y-values become negative; all negative y-values become positive (the graph reflects about the x-axis).

**Example:**
\[
\begin{align*}
\text{Domain: } & x \in [-5, 4] \\
\text{Range: } & y \in [-1, 3] = -[-3, 1].
\end{align*}
\]

\[ y = -f(x) = \begin{cases} 
-1 & \text{if } -5 \leq x \leq -1 \\
(-x) & \text{if } -1 < x < 3 \\
-(3x - 12), & \text{if } 3 \leq x \leq 4 
\end{cases} \]

\[ = \begin{cases} 
-1 & \text{if } -5 \leq x \leq -1 \\
x & \text{if } -1 < x < 3 \\
-3x + 12, & \text{if } 3 \leq x \leq 4 
\end{cases} \]

**Note:** Multiply each \(f(x)\) value by \(-1\).

6. Reflection About the Y-Axis
\[ y = f(-x) \]

**General Properties:**
(I) **Domain:** \(-1 \times \text{Dom}_f\) (Interchange the signs of the endpoints of the Domain of \(f(x)\)).
(II) **Range:** \(\text{Range}_f\) (Range of \(f(x)\) unaffected in this horizontal transformation).
(III) **Symmetry:** Preserves both origin symmetry and y-axis symmetry.
(IV) All positive x-values become negative; all negative x-values become positive (the graph reflects about the y-axis).

**Example:**
\[
\begin{align*}
\text{Domain: } & x \in [-4, 5] = -1 \times [-5, 4] \\
\text{Range: } & y \in [-3, 1].
\end{align*}
\]

\[ y = f(-x) = \begin{cases} 
1 & \text{if } -5 \leq -x \leq -4 \\
(-x) & \text{if } -1 < -x < 3 \\
3(-x) - 12, & \text{if } 3 \leq -x \leq 4 
\end{cases} \]

\[ = \begin{cases} 
1 & \text{if } 1 \leq x \leq 5 \\
x & \text{if } -3 < x < 1 \\
-3x - 12, & \text{if } -4 \leq x \leq -3 
\end{cases} \]

**Note:** Replace EACH \(x\) with \((-x)\), including in the domain statement then simplify.
Supplemental Notes: Transformations of Graphs of Functions

7. Absolute Value of \( f(x) \) (Partial Reflection About the X-Axis)
\[
y = |f(x)|
\]
**General Properties:**
(I) **Domain:** \( \text{Dom}_f \) (Domain is unaffected in vertical transformation)
(II) **Range:** \( |\text{Range}_f| \) (Range runs from smallest non-negative value of Range of \( f(x) \) to greater of absolute value of negative & positive endpoints.)
(III) **Symmetry:** Destroys origin symmetry but preserves y-axis symmetry.
(IV) All negative y-values become positive (negative part of graph reflects about the x-axis).
Y-values are unaffected

**Example:**
\[
y = |f(x)|
\]
\[
\begin{align*}
\text{Domain: } x &\in [-5, 4] \\
\text{Range: } y &\in [0, 3] = |[-3, 1]|
\end{align*}
\]

8. \( f \) of Absolute Value X (Partial Reflection About the Y-Axis)
\[
y = f(|x|) \quad \text{Note that this transformation changes the range of the original function.}
\]
**General Properties:**
(I) **Domain:** The part of domain corresponding to \( x \geq 0 \) is reflected to part corresponding to \( x \leq 0 \). \( \text{E.g.:} \) \([-3, 4]\) becomes \([-4, 4]\), or \((-5,3)\) becomes \([-3, 3]\).
(II) **Range:** Range corresponding to \( x \geq 0 \) (Note that Range IS affected).
(III) **Symmetry:** Destroys origin symmetry creates y-axis symmetry.
(IV) The side of graph left of y-axis is replaced by a reflection of right side about the y-axis. The right side is unaffected (it remains as it was in \( f(x) \)).

**Example:**
\[
y = f(|-x|)
\]
\[
\begin{align*}
\text{Domain: } x &\in [-4, 4] \\
\text{Range: } y &\in [-3, 0]
\end{align*}
\]
Note: Replace each \( f(x) \) value with \( |f(x)| \).
Supplemental Notes: Transformations of Graphs of Functions

Note:

A) In the case of a horizontal transformation, change ALL x-values as indicated by the transformation. For example, take the example function:

\[ y = f(x) = \begin{cases} 
1 & \text{if } -5 \leq x \leq -1 \\
-x & \text{if } -1 < x < 3 \\
3x - 12 & \text{if } 3 \leq x \leq 4
\end{cases} \]

To do the transformation \( f(2x) \), replace all \( x \) values with \( 2x \), and then simplify the expressions as shown below:

\[ y = f(2x) = \begin{cases} 
1 & \text{if } -5 \leq 2x \leq -1 \\
-(2x) & \text{if } -1 < 2x < 3 \\
3(2x) - 12 & \text{if } 3 \leq 2x \leq 4
\end{cases} = \begin{cases} 
1 & \text{if } -2.5 \leq x \leq -0.5 \\
-2x & \text{if } -0.5 < 2x < 1.5 \\
6x - 12 & \text{if } 1.5 \leq x \leq 2
\end{cases} \]

This allows the domain to be automatically adjusted to the correct domain (the new domain is now \([-2.5, 2]\) as indicated in transformation 4.

B) It is best to draw the graph of the transformed function before determining the new domain. A change in \( x \)-values may cause a change in the range values (especially in the case of absolute value transformation). It is easier to see this change after the graph of the new function is drawn.

C) Typically, when doing transformations of functions, it is best to rewrite the function in terms of a fundamental function, draw the fundamental function, then do the transformations from the fundamental function one at a time until the desired graph is reached.

In addition, it is best to express the transformation in the form:

\[ kf(A(x + c)) + B, \] where \( A, B, c, k \) are constants and \( k \neq 0 \neq c \) (\( k \) & \( c \) are non-zero), then do the transformations in the following order:

1. Horizontal stretch (stretch or compress horizontally with respect to \( A \) as appropriate).
2. Vertical stretch (stretch or compress vertically with respect to \( k \) as appropriate).
3. Horizontal shift (shift horizontal \( c \) places as appropriate)
4. Vertical shift (shift vertically \( B \) places as appropriate).

Note: Steps 1 & 2 can be done at the same time. Steps 3 & 4 can be done at the same time.

This approach works for most transformations. It is always good to practice the transformations to recognize how best to do these transformations.