MAC 1105: Quiz 11

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

Find the composite function for the given functions.
1) \( f \circ g \) for \( f(x) = 8x + 13 \) and \( g(x) = 4x - 1 \)
   - A) \( 32x + 51 \)
   - B) \( 32x + 21 \)
   - C) \( 32x + 5 \)
   - D) \( 32x + 12 \)
   \[ f(g(x)) = 8(4x - 1) + 13 = 32x - 8 + 13 = 32x + 5 \]  
   \[ \bigcirc \text{C) } 32x + 5 \]  

Evaluate the expression.
2) \( (g \circ f)(-7) \) when \( f(x) = \frac{x - 8}{5} \) and \( g(x) = 9x + 2 \).
   - A) \(-33\)
   - B) \(183\)
   - C) \(-25\)
   - D) \(-69\) \( \frac{5}{5} \)
   \[ g(f(-7)) = g\left(\frac{-7 - 8}{5}\right) = g\left(-\frac{15}{5}\right) = g(-3) = 9(-3) + 2 = -25 \]  
   \[ f(-7) = \frac{-7 - 8}{5} = -3 \]  
   \[ \bigcirc \text{C) } -25 \]  

Assume the functions are one-to-one. Find the requested inverse.
3) \[
\begin{array}{c|cccc}
  x & -9 & -6 & -4 & 3 & 7 & 9 \\
\hline
  f(x) & -7 & -4 & -2 & 5 & 9 & 11
\end{array}
\]
   \[
\begin{array}{c|cccc}
  x & -7 & -4 & -2 & 5 & 9 & 11 \\
\hline
  g(x) & -13 & -7 & -3 & 11 & 19 & 23
\end{array}
\]
   Find \( (f^{-1} \circ f^{-1})(-2) \) \( f^{-1}(f^{-1}(-2)) = f^{-1}(-4) = -6 \)
   - A) \(-6\)
   - B) \(-2\)
   - C) \(-4\)
   - D) \(-9\)
   \[ \bigcirc \text{A) } -6 \]  

Find the inverse of the relation.
4) \[
\begin{array}{c|cccc}
  x & 6 & 12 & 10 & 8 \\
\hline
  f(x) & 8 & -4 & -3 & -5
\end{array}
\]
   \[
\begin{array}{c|ccccc}
  f:\{ (6, -6), (12, -5), (10, -4), (8, -3) \}
\end{array}
\]
   - A) \{(-5, -6), (-6, 10), (6, 12), (-5, -4)\}
   - B) \{(-6, 6), (-5, 12), (-4, 10), (-3, 8)\}
   - C) \{(-5, -6), (-3, 10), (6, 10), (-5, -4)\}
   - D) \{(6, 6), (12, 12), (-4, 10), (-3, 8)\}
   \[ \bigcirc \text{B) } \{(-6, 6), (-5, 12), (-4, 10), (-3, 8)\} = f^{-1} \]
Quadratic Functions

Summary Notes

A Quadratic Function is one of the form: \( f(x) = ax^2 + bx + c \) where \( a \neq 0 \)
This form is called **Standard Form** of a quadratic equation.

Another form of the equation is based on the graph of the function: \( f(x) = x^2 \), which is a parabola that opens upward, symmetric about the y-axis and has vertex at \((0, 0)\) as shown below:

The second form of a quadratic function indicates how \( f(x) = x^2 \) can be modified to get the graph of \( f(x) = ax^2 + bx + c \). This form is called **Vertex Form**, and looks like this:

\[
f(x) = a(x-h)^2 + k
\]
where \((h, k)\) is the vertex.

This form indicates that the graph of \( f(x) = ax^2 + bx + c \) can be obtained from the graph of \( f(x) = x^2 \) by stretching it \( a \) units, then shifting it \( h \) units to the right and \( k \) units upward (opposite direction if \( a, h \) or \( k \) are negative).

The graph of \( f(x) = a(x-h)^2 + k \) is also symmetric about the line \( x = h \). Thus, \( x = h \) is the **axis of symmetry** of the graph of the function.

**Information about the graph of** \( f(x) = ax^2 + bx + c \)

To graph the function \( f(x) = ax^2 + bx + c \), the following information is useful:

1. **Does the graph open upward or downward?**
   - If \( a > 0 \), then the graph opens upward.
   - If \( a < 0 \), then the graph opens downward.

2. **What is the vertex?**
   - The vertex is the point \((h, k)\), and the coordinates can be obtained as follows:

\[
f(h) = k
\]
Information about the graph of \( f(x) = ax^2 + bx + c \)

To graph the function \( f(x) = ax^2 + bx + c \), the following information is useful:

1. **Does the graph open upward or downward?**
   - If \( a > 0 \), then the graph opens upward.
   - If \( a < 0 \), then the graph opens downward.

2. **What is the vertex?**
   - The vertex is the point \((h, k)\), and the coordinates can be obtained as follows:
     \[
     h = \frac{-b}{2a} \quad \text{and} \quad k = f(h) = f\left(\frac{-b}{2a}\right)
     \]
   - So the vertex of the function \( f(x) = ax^2 + bx + c \) is \( \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right) \).

   - Note that the vertex of the graph is the lowest point (called the minimum point) if it opens upward, and the highest point (maximum point) if it opens downward.

   - Note: To convert from standard form to vertex form calculate \( h \) and \( k \) and substitute into the formula as shown below:
     \[
     \text{Set} \quad h = \frac{-b}{2a} \quad \text{and} \quad k = f(h) = f\left(\frac{-b}{2a}\right), \quad \text{then we can get the same result. That is, we can write} \quad f(x) = ax^2 + bx + c
     \]

   - as \( f(x) = a(x-h)^2 + k \).

3. **What is the axis of symmetry?**
   - The axis of symmetry is \( x = h \). That is, \( x = \frac{-b}{2a} \), which is a vertical line.

4. **What is the y-intercept?**
   - To obtain the y-intercept, set \( x = 0 \), then solve for \( y \). That is, find \( f(0) \). Note that if the function is in standard form, \( f(x) = ax^2 + bx + c \), then the y-intercept is \( c \), because \( f(0) = c \).
   - Note that a quadratic function always has exactly one y-intercept.

\[
\begin{align*}
\text{Example:} & \quad f(x) = -2x^2 - 12x + 7 \\
& \quad a = -2, \quad b = -12, \quad c = 7 \\
& \quad h = \frac{-(-12)}{2(-2)} = -3 \quad \text{and} \quad k = f(-3) = -2(-3)^2 - 12(-3) + 7 = 25
\end{align*}
\]

\[
\begin{align*}
\text{Vertex:} & \quad (-3, 25) \\
\Rightarrow f(x) &= -2(x - (-3))^2 + 25 \\
\Rightarrow f(x) &= -2(x + 3)^2 + 25
\end{align*}
\]

\[
\begin{align*}
\text{At } x &= -3 \\
\Rightarrow f(x) &= -2(0)^2 + 12(0) + 7 = 7
\end{align*}
\]

\[
\begin{align*}
\text{At } x &= -3, \\
\Rightarrow f(x) &= -2(0)^2 - 12(0) + 7 \quad \text{or} \quad (0, 7)
\end{align*}
\]

Quadratic Functions Summary Notes
5. Are there any x-intercepts? If so, what are they?
To determine the x-intercept, set \( f(x) = 0 \), then solve for \( x \). That is, solve the equation: \( ax^2 + bx + c = 0 \)
This is harder than it sounds. The following sections of the notes address the solution of the equation

\[
\text{Set } f(x) = 0 \quad \text{solve for } x.
\]

\[
f(x) = -2x^2 - 12x + 7
\]

\[
-2x^2 - 12x + 7 = 0
\]

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(-2)(7)}}{2(-2)}
\]

\[
x = \frac{12 \pm \sqrt{200}}{-4} \quad \Rightarrow \quad x = \frac{12 + 10\sqrt{2}}{-4} \quad \text{or} \quad x = \frac{12 - 10\sqrt{2}}{-4}
\]

\[
x \approx -6.536 \quad \text{or} \quad x \approx 0.536
\]

\( x \)-intercepts: \((-6.536, 0) \) and \((0.536, 0)\)
The Discriminant

In the quadratic formula, the portion under the square root (that is, \( b^2 - 4ac \)) tells whether you will obtain one real number solution, two real number solutions, or no real number solutions.

\( b^2 - 4ac \) is called the Discriminant of the expression.

The Discriminant indicates the number of solutions as follows:

If \( b^2 - 4ac = 0 \), then there is one real number solution. That is the graph touches the x-axis once.

If \( b^2 - 4ac > 0 \), then there are two real number solutions. That is the graph crosses the x-axis twice.

If \( b^2 - 4ac < 0 \), then there is no real number solution. In fact, there are actually two complex number solutions, which are conjugates of each other. In this case, the graph has no x-intercept. It is either totally above the x-axis (if it opens upward) or totally below the x-axis (if it opens downward).
Ex. 3.1 (p. 329). For #27-38, do the following:
(a) Find the vertex; (b) Find the axis of symmetry; (c) Write the equation in vertex form
(d) Find all intercepts; (e) Tell whether the graph has a maximum or minimum and tell the value.

28.) \( y = x^2 - 2x + 2 \)

\[ h = \frac{-b}{2a} = \frac{-(2)}{2(1)} = -1 \]

\[ k = h^2 - 2(1) + 2 = 1 \]

(a) \text{Vertex: } (1, 1)

(b) \text{Axis: } x = 1

(c) \( y = a(x-h)^2 + k \)

\[ y = (x-1)^2 + 1 \]

\[ a = 1 \]

(d) \text{y-intercept: } y = 0^2 - 2(0) + 2 = 2

\[ (0, 2) \]

No \text{x-intercept: } (x-1)^2 + 1 = 0

\[ (x-1)^2 = -1 \]

\[ x-1 = \pm \sqrt{-1} \]

\[ x = 1 \pm i \]

\[ \text{Not real numbers.} \]

(e) Minimum value is 1 \( y \)-coordinate

\( a > 0 \)
Ex. 3.1 (p. 329). For #27-38, do the following: 
(a) Find the vertex; (b) Find the axis of symmetry; (c) Write the equation in vertex form 
(d) Find all intercepts; (e) Tell whether the graph has a maximum or minimum and tell the value.

30. \( y = 8 + 3x - x^2 \)

\[ y = -x^2 + 3x + 8 \]
\( a = -1 \quad b = 3 \quad c = 8 \)

\( a \neq 0 \)

(a) \( h = \frac{-b}{2a} = \frac{-3}{2(-1)} = \frac{3}{2} \)

\( k = -h^2 + 3h + 8 = \frac{41}{4} \)

\( \text{Vertex: } \left( \frac{3}{2}, \frac{41}{4} \right) \)

(b) \( \text{Axis: } x = \frac{3}{2} \)

c) \( y = -(x - \frac{3}{2})^2 + \frac{41}{4} \)

\( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)
\[ x = \frac{3 \pm \sqrt{3^2 - 4(-1)(8)}}{2(-1)} \]
\[ x = \frac{3 \pm \sqrt{9 + 32}}{-2} \]
\[ x = \frac{3 \pm \sqrt{41}}{-2} \]
\[ x_1 = \frac{-3 + \sqrt{41}}{-2} \approx -3 + 6.4031 \]
\[ x_2 = \frac{-3 - \sqrt{41}}{-2} \approx -3 - 6.4031 \]

Intercepts: \((0, 8), (-1.702, 0), (4.702, 0)\)

(e) **Maximum value of** \(\frac{41}{4}\)
Ex. 3.1 (p. 329). For #13-22, find the vertex form equation of the quadratic function with the given vertex and passing through the given point.

17.) Vertex: (2, 5)  Pt. (3, 7)  
\[ y = a(x-h)^2 + k \]
\[ y = a(x-2)^2 + 5 \]
We need to find \( a \):
Plug in (3, 7):
\[ 7 = a(3-2)^2 + 5 \]
\[ 7 = a + 5 \]
\[ a = 2 \]
Equation: \[ y = 2(x-2)^2 + 5 \]

20.) Vertex: (-3, -2)  Pt. (0, -8)  
\[ y = a(x-h)^2 + k \]
\[ y = a(x+3)^2 - 2 \]
To find \( a \):
\[ -8 = a(0+3)^2 - 2 \]
\[ -8 = 9a - 2 \]
\[ -6 = 9a \]
\[ a = \frac{-6}{9} = \frac{-2}{3} \]
Equation: \[ y = \frac{-2}{3}(x+3)^2 - 2 \]
$y = b^x$

where $b > 0$, $b \neq 1$
We’ve looked at linear and quadratic functions, polynomial functions and rational functions. We are now going to study a new function called **exponential functions**. They are different than any of the other types of functions we’ve studied because the independent variable is in the exponent.

![Graph of exponential function](image)

Let’s look at the graph of this function by plotting some points.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$2^x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>1/2</td>
</tr>
<tr>
<td>-2</td>
<td>1/4</td>
</tr>
<tr>
<td>-3</td>
<td>1/8</td>
</tr>
</tbody>
</table>

Recall what a negative exponent means:

$$f(-1) = 2^{-1} = \frac{1}{2}$$
Compare the graphs $2^x$, $3^x$, and $4^x$

**Characteristics about the Graph of an Exponential Function** \( f(x) = a^x \) where \( a > 1 \)

1. Domain is all real numbers
2. Range is positive real numbers
3. There are no \( x \) intercepts because there is no \( x \) value that you can put in the function to make it = 0
4. The \( y \) intercept is always \((0,1)\) because \( a^0 = 1 \)
5. The graph is always increasing
6. The \( x \)-axis (where \( y = 0 \)) is a horizontal asymptote for \( x \to -\infty \)
$y = 2^x$

All of the transformations that you learned apply to all functions, so what would the graph of $y = 2^x + 3$ look like?

$y = 2^{x-2} - 1$

Reflected over the $x$-axis

Up 1

Right 2

Down 1

Up 3

$y = 1 - 2^x$
Reflected over $y$-axis

$$y = 2^{-x}$$

This equation could be rewritten in a different form:

$$y = 2^{-x} = \frac{1}{2^x} = \left(\frac{1}{2}\right)^x$$

So if the base of our exponential function is between 0 and 1 (which will be a fraction), the graph will be decreasing. It will have the same domain, range, intercepts, and asymptote.

There are many occurrences in nature that can be modeled with an exponential function (we’ll see some of these later this chapter). To model these we need to learn about a special base.
The Base “e” (also called the natural base)

To model things in nature, we’ll need a base that turns out to be between 2 and 3. Your calculator knows this base. Ask your calculator to find $e^1$. You do this by using the $e^x$ button (generally you’ll need to hit the 2nd or yellow button first to get it depending on the calculator). After hitting the $e^x$, you then enter the exponent you want (in this case 1) and push = or enter. If you have a scientific calculator that doesn’t graph you may have to enter the 1 before hitting the $e^x$. You should get 2.718281828

Example for TI-83
If \( a^u = a^v \), then \( u = v \)

This says that if we have exponential functions in equations and we can write both sides of the equation using the same base, we know the exponents are equal.

\[
2^{3x-4} = 8
\]

The left hand side is 2 to the something. Can we re-write the right hand side as 2 to the something?

\[
2^{3x-4} = 2^3
\]

Now we use the property above. The bases are both 2 so the exponents must be equal.

\[
3x - 4 = 3
\]

We did not cancel the 2’s, We just used the property and equated the exponents.

\[
3x = 7
\]

\[
x = \frac{7}{3}
\]

You could solve this for \( x \) now.
Let’s try one more: \(4^x = \frac{1}{8}\)

We could however re-write both the left and right hand sides as 2 to the something.

\[
\left(2^2\right)^x = 2^{-3}
\]

\[
2^{2x} = 2^{-3}
\]

\[
x = -\frac{3}{2}
\]

The left hand side is 4 to the something but the right hand side can’t be written as 4 to the something (using integer exponents)

So now that each side is written with the same base we know the exponents must be equal.

Check:

\[
4^{\left(-\frac{3}{2}\right)} = \frac{1}{8}
\]

\[
\frac{1}{4^{\frac{3}{2}}} = \frac{1}{8}
\]

\[
\frac{1}{\sqrt[2]{(4)^3}} = \frac{1}{8}
\]