MAC 1105: Quiz 16

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question.

Solve the system by the substitution method. If the equations are dependent, write your answer with y being arbitrary.

1) \[
\begin{align*}
\begin{cases}
  x + y &= 6 \quad (i) \\
  y &= 4x - 4 \quad (ii)
\end{cases}
\end{align*}
\Rightarrow (i) : x + 4x - 4 = 6
\]
A) \{(1, 6)\}  
B) \{(2, 4)\}  
C) \{(3, 5)\}  
D) \{(4, 2)\}

Solve the system by the elimination method. If the equations are dependent, write your answer with x being arbitrary.

2) \[
\begin{align*}
\begin{cases}
  x + 4y &= 13 \quad (i) \\
  2x + 3y &= 6 \quad (ii)
\end{cases}
\end{align*}
\]
A) \{(3, 5)\}  
B) \{(-4, 5)\}  
C) \{(-3, 4)\}  
D) \emptyset

Determine whether the system of equations is consistent or inconsistent. If the system is consistent, determine whether the equations are dependent or independent.

3) \[
\begin{align*}
\begin{cases}
  8 &= 9x + 7y \quad (i) \\
  -21y - 27x &= -24 \quad (ii)
\end{cases}
\end{align*}
\]
A) Inconsistent  
B) Dependent  
C) Independent

4) \[
\begin{align*}
\begin{cases}
  x + y &= 2 \quad (i) \\
  x + y &= 5 \quad (ii)
\end{cases}
\end{align*}
\]
A) Inconsistent  
B) Independent  
C) Dependent
Systems of Linear Equations

Three Equations Containing Three Variables

\[ a_{11}x + a_{12}y + a_{13}z = b_1 \]
\[ a_{21}x + a_{22}y + a_{23}z = b_2 \]
\[ a_{31}x + a_{32}y + a_{33}z = b_3 \]

As before, the first two cases are called consistent since there are solutions. The last case is called inconsistent.

The solution will be one of three cases:

1. Exactly one solution, an ordered triple \((x, y, z)\)
2. A dependent system with infinitely many solutions
3. No solution
With two equations and two variables, the graphs were lines and the solution (if there was one) was where the lines intersected. Graphs of three variable equations are planes. Let’s look at different possibilities. Remember the solution would be where all three planes all intersect.

Planes intersect at a point: consistent with one solution INDEPENDENT
Planes intersect in a line: consistent system called dependent with an infinite number of solutions
Three parallel planes: no intersection so system called inconsistent with no solution
No common intersection of all three planes: inconsistent with no solution
We will be doing elimination to solve these systems. It’s like elimination that you learned with two equations and variables but it’s now the problem is bigger.

\[
\begin{align*}
2x + y + z &= 4 \\
-3x + 2y - 2z &= -10 \\
\boxed{x - 2y + 3z &= 7}
\end{align*}
\]

Coefficient is a 1 here so it will be easy to work with

Your first strategy would be to choose one equation to keep that has all 3 variables, but then use that equation to “eliminate” a variable out of the other two. I’m going to choose the last equation to “keep” because it has just \(x\).
\[ 2x + y + z = 4 \quad (i) \]
\[ x - 2y + 3z = 7 \quad (ii) \]
\[ -3x + 2y - 2z = -10 \quad (iii) \]

Keep over here for later use

Now use the third equation multiplied through by whatever it takes to eliminate the \( x \) term from the first equation and add these two equations together. In this case when added to eliminate the \( x \)'s you'd need a -2.

\[-2(x - 2y + 3z = 7) \rightarrow -2x + 4y - 6z = -14 \]
\[ 2x + y + z = 4 \quad (iv) \]

Put this equation up with the other one we kept

\[ 5y - 5z = -10 \]
\[2x + y + z = 4\]  \(\text{(i)}\)
\[-3x + 2y - 2z = -10\]  \(\text{(ii)}\)
\[x - 2y + 3z = 7\]  \(\text{(iii)}\)

\[x - 2y + 3z = 7\]  \(\text{(iv)}\)
\[5y - 5z = -10\]  \(\text{(v)}\)

Now use the third equation multiplied through by whatever it takes to eliminate the \(x\) term from the \textit{second} equation and add these two equations together. In this case when added to eliminate the \(x\)'s you'd need a 3.

\[3(x - 2y + 3z = 7)\]

\[3x - 6y + 9z = 21\]

\[-3x + 2y - 2z = -10\]

\[\Rightarrow -4y + 7z = 11\]

We won’t “keep” this equation, but we’ll use it together with the one we “kept” with \(y\) and \(z\) in it to eliminate the \(y\)’s.
\[ 2x + y + z = 4 \]
\[ -3x + 2y - 2z = -10 \]
\[ x - 2y + 3z = 7 \]
\[ x - 2y + 3z = 7 \]
\[ 5y - 5z = -10 \]
\[ z = 1 \]

So we’ll now eliminate \( y \)’s from the 2 equations in \( y \) and \( z \) that we’ve obtained by multiplying the first by 4 and the second by 5

\[ 5y - 5z = -10 \] \[ \times 4 \]
\[ -4y + 7z = 11 \] \[ \times 5 \]

\[ 20y - 20z = -40 \] \[ \text{(v)} \]
\[ -20y + 35z = 55 \] \[ \text{\textcolor{red}{(v)}} \]

\[ 15z = 15 \]
\[ z = 1 \]

We can add this to the one’s we’ve kept up in the corner
\[ 2x + y + z = 4 \quad (i) \]
\[ -3x + 2y - 2z = -10 \quad (ii) \]
\[ x - 2y + 3z = 7 \quad (iii) \]
\[ x - 2y + 3z = 7 \quad (iv) \]
\[ 5y - 5z = -10 \quad (v) \]
\[ z = 1 \]

Now we are ready to take the equations in the corner and “back substitute” using the equation at the bottom and substituting it into the equation above to find \( y \).

\[ 5y - 5(1) = -10 \]
\[ 5y = -5 \quad y = -1 \]

Now we know both \( y \) and \( z \) we can sub them in the first equation and find \( x \).

\[ x - 2(-1) + 3(1) = 7 \]
\[ x + 5 = 7 \quad x = 2 \]

These planes intersect at a point, namely the point (2, -1, 1). The equations then have this unique solution. This is the ONLY \( x \), \( y \) and \( z \) that make all 3 equations true.
Let’s do another one:

\[ 2x - 3y - z = 0 \quad (i) \]
\[ -x + 2y + z = 5 \quad (ii) \]
\[ 3x - 4y - z = 1 \quad (iii) \]

If we multiply the equation we kept by 3 and add it to the last equation we can eliminate \( x \)'s.

\[ 2(ii) + (i) \]
\[ -2x + 4y + 2z = 10 \]
\[ 2x - 3y - z = 0 \] (that one we’ll keep)
\[ y + z = 10 \quad (iv) \]

\[ 3(ii) + (iii) \]
\[ -3x + 6y + 3z = 15 \]
\[ 3x - 4y - z = 1 \]
\[ 2y + 2z = 16 \quad (v) \]

Now we’ll use the 2 equations we have with \( y \) and \( z \) to eliminate the \( y \)'s.
\[ 2x - 3y - z = 0 \]
\[ -x + 2y + z = 5 \]
\[ 3x - 4y - z = 1 \]
\[ y + z = 10 \]

This means the equations are inconsistent and have no solution. The planes don’t have a common intersection and there is not any \((x, y, z)\) that make all 3 equations true.

\[ 2y + 2z = 16 \]

we’ll multiply the first equation by -2 and add these together

\[ -2y - 2z = -20 \]
\[ 2y + 2z = 16 \]
\[ 0 = -4 \]

Oops---we eliminated the y’s alright but the z’s ended up being eliminated too and we got a false equation.
Let’s do another one:

\[ x + y + 2z = 1 \quad (i) \]
\[ 2x - y + z = 2 \quad (ii) \]
\[ 4x + y + 5z = 4 \quad (iii) \]

If we multiply the equation we kept by -4 and add it to the last equation we can eliminate \( x \)’s.

\[
-2(i) \quad + \quad (ii) \\
-2x - 2y - 4z = -2 \\
\underline{2x - y + z = 2} \\
\underline{\text{we’ll keep this one}} \\
-3y - 3z = 0 \quad (iv)
\]

\[
-4(i) \quad + \quad (iii) \\
-4x - 4y - 8z = -4 \\
\underline{4x + y + 5z = 4} \\
\text{(v) } -3y - 3z = 0
\]

Now we’ll use the 2 equations we have with \( y \) and \( z \) to eliminate the \( y \)’s.
\[
\begin{align*}
    x + y + 2z &= 1 \\
    2x - y + z &= 2 \\
    4x + y + 5z &= 4 \\
    -3y - 3z &= 0 \quad (iv) \\
    -3y - 3z &= 0 \quad (v) 
\end{align*}
\]

This means the equations are consistent and have infinitely many solutions. The planes intersect in a line. To find points on the line we can solve the 2 equations we saved for \( x \) and \( y \) in terms of \( z \).

\[
\begin{align*}
    3y + 3z &= 0 \\
    -3y - 3z &= 0 \\
    \underline{0} &= 0 
\end{align*}
\]

Oops---we eliminated the \( y \)'s alright but the \( z \)'s ended up being eliminated too but this time we got a true equation.
\[ x + y + 2z = 1 \quad (i) \]
\[ 2x - y + z = 2 \quad (ii) \]
\[ 4x + y + 5z = 4 \quad (iii) \]

\[ x + y + 2z = 1 \]
\[ -3y - 3z = 0 \]
\[ z = z \]

First we just put \( z = z \) since it can be any real number. Now solve for \( y \) in terms of \( z \).

Now sub it \(-z\) for \( y \) in first equation and solve for \( x \) in terms of \( z \).

The solution is \((1 - z, -z, z)\) where \( z \) is any real number.

For example: Let \( z \) be 1. Then \((0, -1, 1)\) would be a solution. Notice it works in all 3 equations.

\[(*) \quad \]
\[ x + y + 2z = 1 \]
\[ -3y - 3z = 0 \]
\[ y = -z \]
\[ x - z + 2z = 1 \]
\[ x = 1 - z \]

But so would the point you get when \( z = 2 \) or 3 or any other real number so there are infinitely many solutions.
Ex. 5.2 (p. 526). Solve the system. Identify the system as independent, dependent or inconsistent.

18.) \(x + y + z = 6 \quad (i)\) 
- Eliminate \(z\) using eq \((i)\):
  \[-3(i) + (iii)\]
  \[2x + 3y - z = 5 \quad (ii)\]
  \[3x - 2y + 3z = 8 \quad (iii)\]
  \((i) + (ii): \quad x + y + z = 6 \quad (i)\)
  \[-3x - 3y - 3z = -18\]
  \[2x + 3y - z = 5 \quad (ii)\] +
  \[3x - 2y + 3z = 8 \quad (iii)\]
  \[3x + 4y = 11 \quad (iv)\]
  \[-5y = -10 \quad (v)\]
  \[y = 2\]
  \[3x + 4y = 11\]
  \[3x + 4(2) = 11\]
  \[3x = 3\]
  \[x = 1\]
  \[x + y + z = 6\]
  \[1 + 2 + z = 6\]
  \[z = 3\]

Soln: \((1, 2, 3)\) 
Independent.

32.) \[2x + 4y + 3z = 6 \quad (i)\]
\[x + 2z = -1 \quad (ii)\]
\[2x - 2y + z = -5 \quad (iii)\]
- Eliminate \(y\): \((i) + 2(\text{iii})\)
  \[2x + 4y + 3z = 6 \quad (i)\]
  \[2x - 4y + 2z = -10 \quad (\text{iii})\] +
  \[4x + 5z = -4 \quad (iv)\]
  \[4x - 8z = 4 \quad (ii)\] +
  \[-3z = 0\]
  \[z = 0\]
  \[(i) + 2z = -1\]
  \[x + 2(0) = -1\]
  \[x = -1\]
  \[(iii) x - 2y + z = -5\]
  \[-1 - 2y + 0 = -5\]
  \[-2y = -4\]
  \[y = 2\]

Soln: \((-1, 2, 0)\)
Independent.
34. \( x + y = 9 \) \hspace{0.5cm} (i)

\[ 2y + 3z = 7 \] \hspace{0.5cm} \text{(ii)}

\[ x - 2z = 4 \] \hspace{0.5cm} \text{(iii)}

**Eliminate** \( x \):

\[-1 \text{ (i)} + (\text{iii}) : \]

\[-x - y = -9 \] \hspace{0.5cm} \text{(i)}

\[ x - 2z = 4 \] \hspace{0.5cm} \text{(iii)}

\[ -y - 2z = -5 \] \hspace{0.5cm} \text{(iv)}

\[ 2y + 3z = 7 \] \hspace{0.5cm} \text{(ii)}

\[-y - 2z = -5 \] \hspace{0.5cm} \text{(iv)} \times 2

\[ 2y + 3z = 7 \] \hspace{0.5cm} \text{(ii)}

\[ -2y + 4z = -10 \] \hspace{0.5cm} \text{(iv)}

\[ -2 = -3 \]

\[ z = 3 \]

\[ 2y + 3 \times (3) = 7 \]

\[ 2y = -2 \]

\[ y = -1 \]

**Soln:** \((10, -1, 3)\)

*Independent*
Ex. 6.1 p. 590

64. \( x+y+z = -5 \) (i)
\( 2x-y-z = -4 \) (ii)
\( y+z = -2 \) (iii)

Eliminate \( x \): \(-2(i) + (ii): \)
\[-2x-2y-2z = 10 \] (i)
\[2x - y - z = -4 \] (ii)
\[-3y - 3z = 6 \] (iv)

Let \( y \) be arbitrary:
(iii) \( y+z = -2 \)
\( z = -2-y \)
\( x+y -2-y = -5 \)
\( x = -3 \)

Soln: \((-3, y, -2-y)\)

\[
\begin{align*}
-3y-3z &= 6 \quad \text{(iv)} \\
3y+3z &= -6 \quad \text{(iii)} \\
\hline
0 &= 0 \quad \text{DEPENDENT}
\end{align*}
\]
Matrices

A matrix is a rectangular array of numbers. We subscript entries to tell their location in the array.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Matrices are identified by their size.
\[1 \times 5\]
\[
\begin{bmatrix}
3 & -1 & 5 & 0 & 2
\end{bmatrix}
\]

\[4 \times 1\]
\[
\begin{bmatrix}
-2 \\
6 \\
1 \\
-3
\end{bmatrix}
\]

\[4 \times 4\]
\[
\begin{bmatrix}
2 & -1 & -2 & 4 \\
-1 & 3 & 5 & 7 \\
-2 & 5 & -8 & 9 \\
4 & 7 & 9 & 0
\end{bmatrix}
\]
A matrix that has the same number of rows as columns is called a **square matrix**.
If you take the coefficient matrix and then add a last column with the constants, it is called the augmented matrix. Often the constants are separated with a line.

Augmented matrix $A^# = \begin{bmatrix} 3 & -2 & 5 & 3 \\ -2 & 1 & 4 & -2 \\ 1 & 4 & -7 & 1 \end{bmatrix}$
Elementary Row Operations

Operations that can be performed without altering the solution set of a linear system

1. Interchange any two rows
2. Multiply every element in a row by a nonzero constant
3. Add elements of one row to corresponding elements of another row

We are going to work with our augmented matrix to get it in a form that will tell us the solutions to the system of equations. The three things above are the only things we can do to the matrix but we can do them together (i.e. we can multiply a row by something and add it to another row).
Row Echelon Form

We use elementary row operations to make the matrix look like the one below. The # signs just mean there can be any number here---we don’t care what.

```
[ 1   #   #   #   # ]
[ 0   1   #   #   # ]
[ 0   0   1   1   # ]
```

1. The first non-zero entry (from the left) must be 1.

2. The leading 1 of a row must be to the left of the leading 1 of each subsequent row (the rows below it).

3. All entries below a leading 1 must be 0.

"The Goal"

After we get the matrix to look like our goal, we put the variables back in and use back substitution to get the solutions.
\[
\begin{bmatrix}
1 & 2 & 1 & 1 \\
3 & 5 & 1 & 3 \\
2 & 6 & 7 & 1
\end{bmatrix}
\]

Answer will replace \( r_2 \)

\[-3r_1 + r_2\]

\[
\begin{bmatrix}
-3 & -6 & -3 & -3 \\
3 & 5 & 1 & 3
\end{bmatrix}
\]
\[-3r_1 + r_2 \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 2 & 6 & 7 & 1 \end{bmatrix} \]

\[-2r_1 + r_3 \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 2 & 6 & 7 & 1 \end{bmatrix} \]

\[-3r_1 \quad -3 \quad -6 \quad -3 \quad -3 \]

\[+ r_2 \quad 3 \quad 5 \quad 1 \quad 3 \]

\[0 \quad -1 \quad -2 \quad 0 \]

\[-2r_1 \quad -2 \quad -4 \quad -2 \quad -2 \]

\[+ r_3 \quad 2 \quad 6 \quad 7 \quad 1 \]

Now we’ll use -2 times row 1 added to row 3 to get a 0 there.
\[-3r_1 + r_2 \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 2 & 6 & 7 & 1 \end{bmatrix} \]

\[-2r_1 + r_3 \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 2 & 5 & -1 \end{bmatrix} \]

Now our first column is like our goal.

\[-3r_1 + r_2 \begin{bmatrix} -3 & -6 & -3 & -3 \\ 3 & 5 & 1 & 3 \\ 0 & -1 & -2 & 0 \end{bmatrix} \]

\[-2r_1 + r_3 \begin{bmatrix} -2 & -4 & -2 & -2 \\ 2 & 6 & 7 & 1 \\ 0 & 2 & 5 & -1 \end{bmatrix} \]

Now we’ll use -2 times row 1 added to row 3 to get a 0 there.
Ex. 6.1 (p. 588)

Determine the order of each matrix.

1) \([7]\) 1x1
2) \([1 3 5 9]\) 1x4
3) \([\begin{array}{cc} 3 & 4.3 \\ 2 & -1 \end{array} 7.5 8\]) 2x4

7) \(A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}\)
\(a_{13} = 3\)  \(a_{33} = 11\)
\(a_{31} = 9\)  \(a_{34} = 12\)

8) For matrix \(A\), identify the location of the entry.

(a) 7  \(a_{23}\)  (b) 10  \(a_{32}\)  (c) 4  \(a_{14}\)  (d) 12  \(a_{34}\)

11-16. Write the augmented matrix for the given linear system:

12) \(x_1 + 2x_2 = 7\)  \(3x_1 + 5x_2 = 11\)
14) \(2v - 10 = 3u\)  \(5u + 7 = v\)  \(2x + 3z = -5\)
16) \(x - y = 2\)  \(y - 2z = 7\)

Rewrite:
\([\begin{array}{cc} 1 & 2 \\ 3 & 5 \end{array} 7 11]\)
\([\begin{array}{cc} -3 & 2 \\ 5 & -1 \end{array} 10 -7]\)
\([\begin{array}{ccc} 1 & -1 & 0 \\ 2 & 0 & 3 \end{array} 2 -5 7\])