MAC 2233: Quiz 3

MULTIPLE CHOICE. Choose the one alternative that best completes the statement or answers the question. Write the correct letter choice on the line corresponding to the question.

Solve the problem.

1) Is \( f(x) \) is continuous at \( x = 8 \)?
\[
f(x) = \begin{cases} 
8x - 10 & \text{if } x \leq 8 \\
-16 & \text{if } x > 8
\end{cases}
\]
A) Yes
B) No

\[
\lim_{x \to 8^-} f(x) = 8(8) - 10 = 54
\]
\[
\lim_{x \to 8^+} f(x) = -16
\]
So NO limit

2) Use the graph of the function to determine the limit, if it exists.
\[
\lim_{x \to -1^-} f(x) \quad \lim_{x \to -1^+} f(x)
\]
A) -5; -2
B) -7; -2
C) -7; -5
D) -2; -7

3) \[
\lim_{x \to -2^-} f(x) \quad \lim_{x \to -2^+} f(x) \quad \lim_{x \to -2} f(x)
\]
3) \( \lim_{x \to 2^-} f(x) \) and \( \lim_{x \to 2^+} f(x) \) and \( \lim_{x \to 2} f(x) \)

A) 2; -2; -2
B) 2; -2; Does not exist
C) 2; -2; 2
D) Does not exist; Does not exist; Does not exist
Find the derivative.

4) \( f(x) = 15x - 14 \)  
   Find \( f'(x) \).
   \[ \frac{\text{slope}}{} = 15 \]
   \( \text{B) 15} \)  
   A) -15  
   C) 1  
   D) 15x

For the following function, state the interval(s) for which the function is continuous.

5) \( f(x) = \frac{x^2 - 9}{x + 3} \)
   \[ x \neq -3 \]

   A) \((\infty, \infty)\)  
   C) \((-\infty, -9) \cup (-9, \infty)\)  
   B) \((-\infty, 3) \cup (3, \infty)\)  
   D) \((-\infty, -3) \cup (-3, \infty)\)
Notation for Differentiation

One notation used for derivative is already familiar: The derivative of $f(x)$ is denoted $f'(x)$.

Another notation is called **Leibniz notation**: The derivative of $y$ with respect to $x$ is $\frac{dy}{dx}$.

So $y = f(x) \Rightarrow \frac{dy}{dx} = f'(x)$

Other notation for the derivative (with respect to $x$) of a function $y = f(x)$:

$$f'(x) = \frac{dy}{dx} = \frac{d}{dx} [f(x)] = D_x[y] = y'$$

*We will use primarily the first three of notations.*
Some Rules of Differentiation

**Question:** What is the tangent line to a straight line?

**Answer:** The line itself. That is, a straight line is its own tangent line.

Since the derivative is the slope of the tangent line, then the derivative of a straight line (at any point) is the slope of the line:

\[ f(x) = mx + b \quad \Rightarrow \quad f'(x) = m \]

In particular, \( f(x) = mx \) \quad \Rightarrow \quad f'(x) = m \]

Based on this, since a constant is a horizontal straight line, then the derivative of a constant is zero:

\[ f(x) = c \quad \Rightarrow \quad f'(x) = 0 \]

*Continued on the next slide…*
Some Rules of Differentiation

Examples:

\[ f(x) = 4x - 3 \quad \Rightarrow \quad f'(x) = 4 \]
\[ f(x) = -\frac{1}{2}x + 430 \quad \Rightarrow \quad f'(x) = -\frac{1}{2} \]
\[ f(x) = 46x \quad \Rightarrow \quad f'(x) = 46 \]
\[ f(x) = x \quad \Rightarrow \quad f'(x) = 1 \]
\[ f(x) = 5 \quad \Rightarrow \quad f'(x) = 0 \]
\[ f(x) = \pi \quad \Rightarrow \quad f'(x) = 0 \]
\[ f(x) = 3^{21} \quad \Rightarrow \quad f'(x) = 0 \]

Derivative of a constant is 0.

Continue for more rules…
Some Rules of Differentiation

**Simple Power Rule:** \[ \frac{d}{dx}[x^n] = nx^{n-1} \quad n \text{ is any real number.} \]

**Example:** \[ f(x) = x^5 \implies f'(x) = 5x^{5-1} = 5x^4 \]

**Constant Multiple Rule:** \[ \frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x) \quad c \text{ is a constant.} \]

**Example:** \[ f(x) = 6x^{\frac{1}{3}} \implies f'(x) = 6 \cdot \frac{1}{3} x^{\frac{1}{3}-1} = 2x^{-\frac{2}{3}} \]

**Example:** \[ \frac{d}{dx}[6\sqrt{x}] = \frac{d}{dx}[6 \cdot x^{\frac{1}{2}}] = 6 \cdot \frac{1}{2} x^{\frac{1}{2}-1} = 3x^{-\frac{1}{2}} = \frac{3}{x^{\frac{1}{2}}} = \frac{3}{\sqrt{x}} \]

*Continue for more rules...*
Some Rules of Differentiation

**Sum and Difference Rules:**

\[
\frac{d}{dx} \left[ f(x) + g(x) \right] = f'(x) + g'(x)
\]

“Derivative of a sum is the sum of the derivatives.”

\[
\frac{d}{dx} \left[ f(x) - g(x) \right] = f'(x) - g'(x)
\]

“Derivative of a difference is the difference of the derivatives.”

**Example:**

\[
f(t) = 4t^{\frac{4}{3}} - 3t + \frac{8}{t^2} + 5 = 4t^{\frac{4}{3}} - 3t + 8t^{-2} + 5
\]

\[
f'(t) = \frac{d}{dt} \left[ 4t^{\frac{4}{3}} \right] - \frac{d}{dt} \left[ 3t \right] + \frac{d}{dt} \left[ 8t^{-2} \right] + \frac{d}{dt} \left[ 5 \right]
\]

\[
f'(t) = \frac{16}{3} t^{\frac{1}{3}} - 3 - 16t^{-3} + 0 = \frac{16}{3} t^{\frac{1}{3}} - 3 - \frac{16}{t^3}
\]

Continue for more rules…
Find the derivative of the function.

<table>
<thead>
<tr>
<th>6.</th>
<th>( f(x) = -2 )</th>
<th>( f'(x) = 0 )</th>
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<tbody>
<tr>
<td>8.</td>
<td>( g(x) = 3x - 1 )</td>
<td>( g'(x) = 3 )</td>
</tr>
<tr>
<td>12.</td>
<td>( y = x^3 - 9x^2 + 2 )</td>
<td>( \frac{dy}{dx} = 3x^2 - 18x )</td>
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<tr>
<td>16.</td>
<td>( h(x) = \frac{x^{5/2}}{2} )</td>
<td>( h'(x) = \frac{5}{2}x^{3/2} )</td>
</tr>
<tr>
<td>26.</td>
<td>( y = \frac{4x}{x^{-3}} )</td>
<td>( y = 4x^4 ) ( \int \frac{dy}{dx} = 16x^3 )</td>
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</table>
34.) \[ f(x) = x^2 - 3x - 3x^{-2} + 5x^{-3} \]
   \[ f'(x) = 2x - 3 + 6x^{-3} - 15x^{-4} \]

36.) \[ f(x) = x^2 + 4x + \frac{1}{x} \]
   \[ f(x) = x^2 + 4 + x^{-1} \]
   \[ f'(x) = 2x + 4 - x^{-2} \]
   \[ f''(x) = 2x + 4 - \frac{1}{x^2} \]

40.) \[ f(x) = (3x^2 - 5x)(x^2 + 2) \]
   \[ f(x) = 3x^4 + 6x^2 - 5x^3 - 10x \]
   \[ f'(x) = 12x^3 + 12x - 15x^2 - 10 \]

44.) \[ f(x) = -6x^3 + 3x^2 - 2x + 1 \]
   \[ f'(x) = \frac{-6x^3 + 3x^2 - 2x + 1}{x} \]
   \[ f(x) = -6x^2 + 3x - 2 + x^{-1} \]
   \[ f'(x) = -12x + 3 - 0 - x^{-2} \]
   \[ f''(x) = -12x + 3 - \frac{1}{x^2} \]

46.) \[ f(x) = x^{\frac{1}{3}} - 1 \]
   \[ f'(x) = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}} \]
Ex. 2.2: Find the derivative at the given point.

28. \( f(t) = 4 - \frac{4}{3} t \) at \((\frac{1}{2}, \frac{4}{3})\)

\[ f(t) = 4 - \frac{4}{3} t^{-1} \]
\[ f'(t) = 0 + \frac{4}{3} t^{-2} \]
\[ f'(\frac{1}{2}) = \frac{4}{3} \left(\frac{1}{2}\right)^{-2} = \frac{4}{3} \cdot \left(\frac{2}{1}\right)^2 = \frac{16}{3} \]

30. \( y = 3x(x^2 - \frac{2}{x}) \) at \((2, 18)\)

\[ y = 3x^3 - 6 \]
\[ \frac{dy}{dx} = 9x^2 \Rightarrow \frac{dy}{dx} \bigg|_{x=2} = 9 \cdot 2^2 = 36 \]

32. \( f(x) = 3(5-x)^2 \) at \((5, 0)\)

\[ f(x) = 3(5-x)(5-x) \]
\[ f(x) = 3(25 - 5x - 5x + x^2) \]
\[ f(x) = 75 - 30x + 3x^2 \]
\[ f'(x) = 0 - 30 + 6x \]
\[ f'(5) = -30 + 6(5) = 0 \]
Ex. 2.2: Find the equation of the tangent line at the given point.

**Note:** 1. To find the equation of the tangent line we need the slope and a point.
2. The slope of the tangent line is the derivative at the point.

48.) \( y = x^3 + x \) at \((-1, -2)\)

\[
m = \left. \frac{dy}{dx} \right|_{x = -1}
\]
\[
\frac{dy}{dx} = 3x^2 + 1
\]
\[
m = 3(-1)^2 + 1 = 4
\]
\[
y - y_1 = m(x - x_1)
\]
\[
y - (-2) = 4(x - (-1))
\]
\[
y + 2 = 4x + 4
\]
\[
y = 4x + 2
\]

50.) \( f(x) = \frac{1}{\sqrt[3]{x^2}} \) at \((-1, 2)\)

\[
f(x) = \frac{1}{x^{2/3}}
\]
\[
f'(x) = -\frac{2}{3}x^{-5/3} - 1
\]
\[
m = f'(-1) = -\frac{2}{3}(-1)^{-5/3} - 1
\]
\[
m = -\frac{2}{3}(-1)^{-5/3} - 1 = -\frac{1}{3}
\]
\[
y - y_1 = m(x - x_1)
\]
\[
y - 2 = -\frac{1}{3}(x - (-1))
\]
\[
y - 2 = -\frac{1}{3}x - \frac{1}{3}
\]
\[
\Rightarrow y = -\frac{1}{3}x + \frac{5}{3}
\]
Some Rules of Differentiation

**Product Rule:**

\[
\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)
\]

In abbreviated form (which may be easier to remember):

\[
\frac{d}{dx}[f \cdot g] = f' \cdot g + f \cdot g'
\]

Read: “Derivative of f times g equals f-prime g plus f g-prime.”

**Example:** Differentiate \( y = \left( x^3 - 3x \right) \left( 2x^2 + 3x + 5 \right) \)

**Step 1:** Identify \( f(x) \) and \( g(x) \)

\( f(x) \) \hspace{1cm} \( g(x) \)

**Step 2:** Determine \( f'(x) \) and \( g'(x) \)

\[
f(x) = x^3 - 3x \quad \Rightarrow \quad f'(x) = 3x^2 - 3
\]

\[
g(x) = 2x^2 + 3x + 5 \quad \Rightarrow \quad g'(x) = 4x + 3
\]

Continued on the next slide...
Some Rules of Differentiation

Product Rule Example (Cont’d):

Step 3: Substitute the $f(x)$, $g(x)$, $f'(x)$ and $g'(x)$ into the Product Rule formula, and simplify:

\[
\frac{dy}{dx} = \left(3x^2 - 3\right)\left(2x^2 + 3x + 5\right) + \left(x^3 - 3x\right)(4x + 3)
\]

\[
= f'(x)g(x) + f(x)g'(x)
\]

\[
\frac{dy}{dx} = 6x^4 + 9x^3 + 15x^2 - 6x^2 - 9x - 15
\]

\[+ 4x^4 + 3x^3 - 12x^2 - 9x\]

\[
= 10x^4 + 12x^3 - 3x^2 - 18x - 15
\]

Continue for more rules…
Some Rules of Differentiation

**Quotient Rule:**
\[
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}
\]

*In abbreviated form (which may be easier to remember):*
\[
\frac{d}{dx} \left[ \frac{f}{g} \right] = \frac{f' \cdot g - f \cdot g'}{g^2}
\]

**Read:** “Derivative of \( f \) times \( g \) equals \( f \)-prime \( g \) minus \( f \) \( g \)-prime divided by \( g \) squared.”

**Example:** Differentiate
\[
y = \frac{x^3 + 3x + 2}{x^2 - 1}
\]

**Step 1:** Identify \( f(x) \) and \( g(x) \)
- \( f(x) = x^3 + 3x + 2 \)
- \( g(x) = x^2 - 1 \)

**Step 2:** Determine \( f'(x) \) and \( g'(x) \)
- \( f'(x) = 3x^2 + 3 \)
- \( g'(x) = 2x \)

Continued on the next slide…
Some Rules of Differentiation

**Quotient Rule Example (Cont’d):**

**Step 3:** Substitute the \( f(x), g(x), f'(x) \) and \( g'(x) \) into the Quotient Rule formula, and simplify:

\[
\frac{dy}{dx} = \frac{\left(3x^2 + 3\right) \cdot \left(x^2 - 1\right) - \left(x^3 + 3x + 2\right) \cdot \left(2x\right)}{\left(x^2 - 1\right)^2} \cdot \left[ g(x) \right]^2
\]

\[
\frac{dy}{dx} = \frac{\left(3x^4 - 3x^2 + 3x^2 - 3\right) - \left(2x^4 + 6x^2 + 4x\right)}{\left(x^2 - 1\right)^2}
\]

\[
\frac{dy}{dx} = \frac{x^4 - 6x^2 - 4x - 3}{\left(x^2 - 1\right)^2}
\]

*Continue for more rules...*
Ex. 2.4 (p. 128). Find the derivative of the function at the given point.

\[ f(x) = (x^2 + 1)(2x + 5) \]

at \((-1, 6)\)

\[ f(x) = 2x^3 + 5x^2 + 2x + 5 \]

\[ f'(x) = 6x^2 + 10x + 2 \]

\[ f'(-1) = 6(-1)^2 + 10(-1) + 2 = -2 \]

8) \[ h(x) = \frac{x^2}{x+3} \]

at \((-1, \frac{1}{2})\)

**Quotient Rule:**

\[ h'(x) = \frac{u'v - uv'}{v^2} \]

\[ u = x^2 \Rightarrow u' = 2x \]

\[ v = x+3 \Rightarrow v' = 1 \]

\[ h'(x) = 2x \left(\frac{x+3}{x+3} - \frac{x^2(1)}{(x+3)^2}\right) = \frac{2x^2 + 6x - x^2}{(x+3)^2} = \frac{x^2 + 6x}{(x+3)^2} \]

\[ h'(-1) = \frac{(-1)^2 + 6(-1)}{(-1+3)^2} = -\frac{5}{4} \]

12) \[ f(x) = \frac{x+1}{x-1} \]

at \((2, 3)\)

\[ f'(x) = \frac{u'v - uv'}{v^2} \]

\[ u = x+1 \Rightarrow u' = 1 \]

\[ v = x-1 \Rightarrow v' = 1 \]

\[ f'(x) = \frac{1(x-1) - 1(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2} \]

\[ f'(2) = \frac{-2}{(2-1)^2} = -2 \]

14) \[ g(x) = \frac{4x - 5}{x^2 - 1} \]

at \((0, 5)\)

\[ g'(x) = \frac{u'v - uv'}{v^2} \]

\[ u = 4x-5 \Rightarrow u' = 4 \]

\[ v = x^2 - 1 \Rightarrow v' = 2x \]

\[ g'(x) = \frac{4(x^2-1) - (4x-5)(2x)}{(x^2-1)^2} = \frac{4(x^2-1) - (4(x-5)(2x)}{(x^2-1)^2} \]

\[ g'(0) = \frac{4(0^2-1) - (4(0)-5)(2(0))}{(0^2-1)^2} = \frac{-4}{1} = -4 \]
Some Rules of Differentiation

**Chain Rule:**
The **Chain Rule** is the differentiation of composition of functions.

Recall that a composition of functions is a “function of a function.” The output of one function (the “inner” function) is the input of the other function (the “outer” function), as indicated below.

\[ x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x)) \]

**Notation:** \[ f \circ g(x) = f(g(x)) \]

Continued on the next slide…
Some Rules of Differentiation

**Chain Rule:**

\[ \frac{d}{dx} \left[ f(g(x)) \right] = f'(g(x)) \cdot g'(x) \]

“The derivative of a composition of functions is equal to the derivative of the outer function with respect to the inner function (that is, without changing the inner) multiplied by the derivative of the inner function.”

**Example:**

\[ y = \left( x^2 - 2x + 3 \right)^3 \]

\[ \frac{dy}{dx} = 3 \left( x^2 - 2x + 3 \right)^2 \cdot (2x - 2) \]

**Outer function**

\[ f(x) = x^3 \]

**Derivative of inside**

**Derivative of outside**
Some Rules of Differentiation

Additional Chain Rule Notation: \[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \]

Example: If \( y = u^{\frac{4}{3}} \) \& \( u = x^2 + 1 \), then:

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left( \frac{4}{3} u^{\frac{1}{3}} \right) \cdot (2x)
\]

\[
= \frac{4}{3} \left( x^2 + 1 \right)^{\frac{1}{3}} \cdot (2x) = \frac{8x}{3} \left( x^2 + 1 \right)^{\frac{1}{3}}
\]

Derivative:

\( y = u^{\frac{4}{3}} \Rightarrow y = (x^2 + 1)^{\frac{4}{3}} \)

\[
\frac{dy}{dx} = \frac{4}{3} \left( x^2 + 1 \right)^{\frac{1}{3}} \cdot (2x)
\]
Some Rules of Differentiation

**Special Case of the Chain Rule – The General Power Rule:**

If \( y = [f(x)]^n \), then \( \frac{dy}{dx} = n \cdot [f(x)]^{n-1} \cdot f'(x) \)

**Example:** Differentiate \( s(x) = \frac{1}{\sqrt{x^2 - 3x + 4}} \)

**Solution:**

\[
\begin{align*}
s(x) &= \frac{1}{\sqrt{x^2 - 3x + 4}} = \frac{1}{\left(x^2 - 3x + 4\right)^{\frac{1}{2}}} = \left(x^2 - 3x + 4\right)^{-\frac{1}{2}} \\
\frac{ds}{dx} &= -\frac{1}{2} \left(x^2 - 3x + 4\right)^{-\frac{3}{2}} \cdot (2x - 3) \\
\frac{ds}{dx} &= -\frac{1}{2} \cdot \frac{1}{\left(x^2 - 3x + 4\right)^{\frac{3}{2}}} \cdot \frac{(2x - 3)}{1} = -\frac{(2x - 3)}{2 \sqrt{(x^2 - 3x + 4)^3}}
\end{align*}
\]
Higher-Order Derivatives

Technically, what we have been referring to as the derivative is actually the \textit{first derivative}. That is, it is the function obtained when we differentiate a function once.

If we differentiate again, the result is called the \textit{second derivative}. That is, \textit{the second derivative is the derivative of the first derivative}.

Subsequent derivatives are named similarly. For example, the seventh derivative of a function is obtained by taking derivatives seven times (\textit{the derivative of the derivative of the derivative of the derivative of the derivative of the derivative of the derivative of the function}).
Higher Order Derivatives

Notation: Suppose \( y = f(x) \)

First Derivative: \( \frac{dy}{dx} = f'(x) \)

Second Derivative: \( \frac{d^2 y}{dx^2} = f''(x) \)

Third Derivative: \( \frac{d^3 y}{dx^3} = f'''(x) \)

Fourth Derivative: \( \frac{d^4 y}{dx^4} = f^{(4)}(x) \)

\[ \vdots \]

\( n^{th} \) Derivative: \( \frac{d^n y}{dx^n} = f^{(n)}(x) \)
Higher Order Derivatives

Example:
Find \( f'' \left( \frac{1}{2} \right) \) if \( f(t) = \sqrt{2t + 3} \)

Solution:
\[
f(t) = \sqrt{2t + 3} = (2t + 3)^{\frac{1}{2}}
\]
\[
\Rightarrow f'(t) = \frac{1}{2} (2t + 3)^{-\frac{1}{2}} \cdot 2 = (2t + 3)^{-\frac{1}{2}}
\]
\[
\Rightarrow f''(t) = -\frac{1}{2} (2t + 3)^{-\frac{3}{2}} \cdot 2 = -(2t + 3)^{-\frac{3}{2}}
\]
\[
\Rightarrow f'''(t) = -\left( -\frac{3}{2} \right) (2t + 3)^{-\frac{5}{2}} \cdot 2 = 3(2t + 3)^{-\frac{5}{2}}
\]
\[
\Rightarrow f''' \left( \frac{1}{2} \right) = 3 \left( 2 \left( \frac{1}{2} \right) + 3 \right)^{-\frac{5}{2}} = 3(4)^{-\frac{5}{2}} = \frac{3}{32}
\]