Find the equation (in slope-intercept form) for the tangent to the curve at the given point.

13) \( f(x) = \sqrt{x + 7}, \ (2, 3) \)

\[
f(x) = (x + 7)^{\frac{1}{2}}
\]

\[
f'(x) = \frac{1}{2} (x + 7)^{-\frac{1}{2}} (1)
\]

\[
m = f'(2) = \frac{1}{2} (2 + 7)^{-\frac{1}{2}} = \frac{1}{2} \cdot 9^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}
\]

\[
y - y_1 = m(x - x_1)
\]

\[
y - 3 = \frac{1}{6} (x - 2)
\]

\[
y - 3 = \frac{1}{6}x - \frac{1}{3}
\]

\[
y = \frac{1}{6}x - \frac{1}{3} + \frac{3}{1}
\]

Equation: \( y = \frac{1}{6}x + \frac{8}{3} \)

Find numbers \( a \) and \( b \), so that \( f \) is continuous at every point.
Find numbers $a$ and $b$, so that $f$ is continuous at every point.

14) \[ f(x) = \begin{cases} 
  x^2, & x < -5 \\
  ax + b, & -5 \leq x \leq -3 \\
  x + 12, & x > -3 
\end{cases} \]

\[ \lim_{x \to -5^-} f(x) = \lim_{x \to -5^-} (-5a + b) = -5a + b = 25 \] (i)

\[ \lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (-3a + b) = -3a + b = 9 \] (ii)

\[ -5a + b = 25 \]
\[ -3a + b = 9 \]
\[ \begin{cases} 
  -5a + b = 25 \\
  3a - b = -9 
\end{cases} \]
\[ -2a = 16 \]
\[ a = -8 \]

\[ -3(-8) + b = 9 \]
\[ 24 + b = 9 \]
\[ b = -15 \]

\[ a = -8 \quad b = -15 \]
Evaluate the second derivative of the function for the given value of $x$. Give exact answer.

15) $f(x) = 4(1 + 3x)^5; \ x = \frac{1}{3}$

$f'(x) = 20(1 + 3x)^4 \cdot 3$

$f'(x) = 60(1 + 3x)^4$

$f''(x) = 240(1 + 3x)^3 \cdot 3$

$f''(x) = 720(1 + 3x)^3$

$f''(\frac{1}{3}) = 720 \left(1 + 3 \left(\frac{1}{3}\right)\right)^3$

$= 720 \left(2\right)^3$

$= 5760$
16) Find all points of the graph of \( y = 5x^2 + 5x \) whose tangent lines are parallel to the line \( y - 55x = 0 \).

\[
\frac{dy}{dx} = 10x + 5 = 55
\]

\[
10x = 50 \\
x = 5
\]

\[
y = 5(5)^2 + 5(5) = 150
\]

Point(s): \((5, 150)\)
Find \( \frac{dy}{dx} \).

17) \( y = \left( \frac{2x + 4}{x - 4} \right)^4 \)

**METHOD 1:**

\[
\frac{dy}{dx} = 4 \left( \frac{2x+4}{x-4} \right)^3 \cdot \frac{d}{dx} \left[ \frac{2x+4}{x-4} \right]
\]

\[
= \frac{U'V - UV'}{v^2}
\]

\[ U = 2x-4 \quad V = x-4 \]
\[ U' = 2 \quad V' = 1 \]

\[
\frac{dy}{dx} = 4 \left( \frac{2x+4}{x-4} \right)^3 \cdot \frac{2(x-4) - (2x+4)(1)}{(x-4)^2}
\]

\[ = 4 \left( \frac{2x+4}{x-4} \right)^3 \cdot \frac{-12}{(x-4)^2} = \frac{-48}{(x-4)^2} \left( \frac{2x+4}{x-4} \right)^3 = \frac{-48(2x+4)^3}{(x-4)^5} \]
Ex. 2.6 (p. 145): Find the second derivative

2.) \( f(x) = 3x - 1 \)
   \[ f'(x) = 3 \]
   \[ f''(x) = 0 \]

10.) \( f(x) = x^{3/2} \)
   \[ f'(x) = \frac{3}{2} x^{1/2} \]
   \[ f''(x) = \frac{3}{4} x^{-1/2} \]

14.) \( h(x) = x^3(x^2-2x+1) \)
   \[ h'(x) = 3x^2 - 2x^3 + x \]
   \[ h''(x) = 6x^2 - 6x^3 + 1 \]

6.) \( f(x) = 4(x^2-1)^2 \)
   \[ f'(x) = 8(x^2-1)(2x) \]
   \[ f'(x) = 16x(x^2-1) \]
   \[ f''(x) = 16x^3 - 16x \]
   \[ f''(x) = 48x^2 - 16 \]

12.) \( g(t) = -\frac{4}{(t+2)^2} \)
   \[ g'(t) = -4(t+2)^{-3} \]
   \[ g''(t) = -24(t+2)^{-4} \]

Find third derivative:

20.) \( f(x) = \frac{1}{x} \)
   \[ f(x) = x^{-1} \]
   \[ f'(x) = -x^{-2} \]
   \[ f''(x) = 2x^{-3} \]
   \[ f'''(x) = -6x^{-4} = \frac{-6}{x^4} \]

24.) \( f(t) = \sqrt{2t+3} \)
   Find \( f'''(\frac{1}{2}) \)
   \[ f(t) = (2t+3)^{1/2} \]
   \[ f'(t) = \frac{1}{2} (2t+3)^{-1/2} \]
   \[ f''(t) = \frac{-1}{4} (2t+3)^{-3/2} \]
   \[ f'''(t) = \frac{3}{8} (2t+3)^{-5/2} \]
   \[ f'''(\frac{1}{2}) = \frac{3}{8} (2(\frac{1}{2})+3)^{-5/2} = \frac{3}{32} \]
Implicit Differentiation

Recall the Chain Rule:
\[
\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x)
\]

Recall also that the Chain Rule applies to a composition of functions: \(f(g(x))\).

Now, examine the following example:

Suppose \(y = x^5 - 3x^2 + 7\) \(\Rightarrow\) \(\frac{dy}{dx} = 5x^4 - 6x\)

Then \(y^3 = (x^5 - 3x^2 + 7)^3\)

What is the derivative of \(y^3 = (x^5 - 3x^2 + 7)^3\) ?

Ans.: \(\frac{d}{dx}[y^3] = \frac{d}{dx}[(x^5 - 3x^2 + 7)^3]\) \(\ldots\) using the Chain Rule…

Continued on the next slide…
Implicit Differentiation

Example continued:
\[
\frac{d}{dx} [y^3] = \frac{d}{dx} \left[ (x^5 - 3x^2 + 7)^3 \right]
\]

...using the Chain Rule:
\[
= 3 \left( x^5 - 3x^2 + 7 \right)^2 \cdot \frac{d}{dx} [x^5 - 3x^2 + 7] \\
= 3 \left( x^5 - 3x^2 + 7 \right)^2 \cdot (5x^4 - 6x) \\
= 3y^2 \cdot \frac{dy}{dx}
\]

So...
\[
\frac{d}{dx} [y^3] = 3y^2 \cdot \frac{dy}{dx}
\]

Note: ➢ The derivative resembles a regular derivative
➢ An additional \( dy/dx \) is multiplied in the derivative. Why?
➢ We use the Chain Rule to do the derivative.
Implicit Differentiation

A Loose Description of Implicit Differentiation:

\[
\frac{d}{dx} [f(y)] = f'(y) \cdot \frac{dy}{dx}
\]

To differentiate a function of \( y \) with respect to \( x \):

- Differentiate the function as usual (in terms of \( y \)), then
- Multiply by \( \frac{dy}{dx} \).

Note: When differentiating keep the following in mind:

- Always differentiate BOTH SIDES of the equation with respect to the same variable.
- The variable that we differentiate with respect to occurs in the denominator of the derivative expression. For example, if we are seeking \( \frac{dy}{dx} \), then differentiate with respect to \( x \). If we are seeking \( \frac{dV}{dt} \), then differentiate with respect to \( t \).
Implicit Differentiation

**Example:**

Find \( \frac{dy}{dx} \) for \( x^2 - y^3 = 3 \)

\[
\frac{d}{dx}[x^2 - y^3] = \frac{d}{dx}[3]
\]

Note that we differentiate both sides with respect to \( x \).

\[
\frac{d}{dx}[x^2] - \frac{d}{dx}[y^3] = \frac{d}{dx}[3]
\]

**Steps:**

- Differentiate both sides with respect to \( x \). Use the sum/difference rule where necessary.

- Determine whether the term we differentiate contains \( x \) or \( y \). If it is a function of \( x \), then regular derivatives (since we differentiate with respect to \( x \)). If it is a function of a variable other than \( x \), (\( y \) in this case), then it is implicit differentiation.
Implicit Differentiation

Example (Cont’d):

\[
\frac{d}{dx}[x^2] - \frac{d}{dx}[y^3] = \frac{d}{dx}[3]
\]

\[
2x - 3y^2 \frac{dy}{dx} = 0
\]

\[
= 2x - 2x
\]

\[
= 3y^2 \frac{dy}{dx} = \frac{2x}{3y^2}
\]

\[
\frac{dy}{dx} = \frac{2x}{3y^2}
\]

Steps:

- Differentiate each term using the appropriate rules of differentiation.
  Remember, for implicit differentiation, differentiate as usual, but multiply by \(dy/dx\) at the end.

- Solve the equation for \(dy/dx\).
Implicit Differentiation

Another Example:

Find \( \frac{dy}{dx} \) for \( x^2y - y^2x = -6 \)

at \((2,-1)\).

\[
\frac{d}{dx} \left[ x^2y \right] - \frac{d}{dx} \left[ y^2x \right] = \frac{d}{dx} \left[ -6 \right]
\]

For \( \frac{d}{dx} \left[ x^2y \right] \),

let \( f(x) = x^2 \) & \( g(x) = y \)

So \( \frac{d}{dx} \left[ x^2y \right] = f'(x) \cdot g(x) + f(x) \cdot g'(x) \)

Steps:

- Differentiate both sides with respect to \( x \). Use the sum/difference rule where necessary.

- To differentiate a product, use the Product Rule. Be sure to put all \( x \) in one fxn and all \( y \) in the other.

One fxn in terms of \( x \)

One fxn in terms of \( y \)

Continued on next slide…
Implicit Differentiation

Example 1 (Cont’d):
\[
\frac{d}{dx}[x^2y] - \frac{d}{dx}[y^2x] = \frac{d}{dx}[-6]
\]

Steps:

➢ Note that both parts of the product are in fxns of x: \(f(x)\) & \(g(x)\).

➢ When doing each differentiation, be sure to identify whether you need to do implicit differentiation or regular differentiation.

For \(\frac{d}{dx}[x^2y]\),

let \(f(x) = x^2\) & \(g(x) = y\)

\(f(x) = x^2 \Rightarrow f'(x) = 2x\)

Regular differentiation, since \(f(x)\) is a fxn of \(x\) and we differentiate \(dx\).

\(g(x) = y \Rightarrow g'(x) = 1 \cdot \frac{dy}{dx} = \frac{dy}{dx}\)

Implicit differentiation, since \(f(x)\) is a fxn of \(y\) and we differentiate \(dx\).

Continued on next slide…
**Implicit Differentiation**

**Example 1 (Cont’d):**

\[ f(x) = x^2 \implies f'(x) = 2x \]

\[ g(x) = y \implies g'(x) = 1 \cdot \frac{dy}{dx} = \frac{dy}{dx} \]

\[
\frac{d}{dx} [x^2 y] = f'(x) \cdot g(x) + f(x) \cdot g'(x)
\]

\[
= 2x \cdot y + x^2 \cdot \frac{dy}{dx} = 2xy + x^2 \frac{dy}{dx}
\]

**Steps:**

- **Complete the Product Rule.** Be careful to substitute carefully.

- **Do the same for all products.**

**Similarly,** for \( \frac{d}{dx} [y^2 x] \)

**let** \( f(x) = y^2 \)** &** \( g(x) = x \)**

\[ \Rightarrow f'(x) = 2y \frac{dy}{dx} \quad g'(x) = 1 \]

\[
\frac{d}{dx} [y^2 x] = 2y \frac{dy}{dx} \cdot x + y^2 \cdot 1 = 2xy \frac{dy}{dx} + y^2
\]

**Regular differentiation,** since \( g(x) \) is a fn of \( x \) and we differentiate \( dx \).

**Implicit differentiation,** since \( f(x) \) is a fn of \( y \) and we differentiate \( dx \).

*Continued on next slide…*
**Example 1 (Cont’d):**

\[
\frac{d}{dx}[x^2 \cdot y] - \frac{d}{dx}[y^2 \cdot x] = \frac{d}{dx}[-6]
\]

\[
\left(2xy + x^2 \frac{dy}{dx}\right) - \left(2xy \frac{dy}{dx} + y^2\right) = 0
\]

\[
\left(2(2)(-1) + (2^2) \frac{dy}{dx}\right) - \left(2(2)(-1) \frac{dy}{dx} + (-1)^2\right) = 0
\]

\[-4 + 4 \frac{dy}{dx} + 4 \frac{dy}{dx} - 1 = 0\]

\[8 \frac{dy}{dx} = 5 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{5}{8}\]

**Steps:**

- Substitute all derivatives into the original equation.
- Since we wish to find \(dy/dx\) at point \((2, -1)\), substitute \(x = 2\) & \(y = -1\), then solve for \(dy/dx\).

**Answer**
Ex. 2.7  p. 152
Find \( \frac{dy}{dx} \):

4) \( 4x^2y - \frac{3}{y} = 0 \)

\[ 4x^2y - 3y^{-1} = 0 \]

\[ \frac{d}{dx} \left[ 4x^2y \right] = u'v + uv' \]

\( u(x) = 4x^2 \Rightarrow u'(x) = 8x \)
\( v(x) = y \Rightarrow v'(x) = \frac{dy}{dx} \)

\[ \frac{d}{dx} \left[ 4x^2y \right] = 8xy + 4x^2 \frac{dy}{dx} \]

So \[ \frac{d}{dx} \left[ 4x^2y - 3y^{-1} \right] = \frac{d}{dx} [0] \]

\[ 8xy + 4x^2 \frac{dy}{dx} + 3y^{-2} \frac{dy}{dx} = 0 \]

Factor \( \frac{dy}{dx} \)

\[ \frac{dy}{dx} \left( 4x^2 + 3y^{-2} \right) = -8xy \]

\[ \frac{dy}{dx} = \frac{-8xy}{4x^2 + 3y^{-2}} \]
Ex. 2.7  p. 152
Find \( \frac{dy}{dx} \):

8.) \( 2x y^3 - x^2 y = 2 \)

Differentiate \( \frac{d}{dx} \):

\[
(u'v + uv') - (p'q + pq') = \frac{d}{dx} \left[ 2 \right]
\]

\( u(x) = 2x \) \quad \( p(x) = x^2 \)

\( u'(x) = 2 \) \quad \( p'(x) = 2x \)

\( v(x) = y^3 \) \quad \( q(x) = y \)

\( v'(x) = 3y^2 \frac{dy}{dx} \) \quad \( q'(x) = \frac{dy}{dx} \)

\[
2 y^3 + 6xy^2 \frac{dy}{dx} - 2xy - x \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} \left( 6xy^2 - x^2 \right) = 2xy - 2y^3
\]

\[
\frac{dy}{dx} = \frac{2xy - 2y^3}{6xy^2 - x^2}
\]
Ex. 2.7  p. 152

Find \( \frac{dy}{dx} \):

10.) \( \frac{xy - y^2}{y - x} = 1 \) multiply by \( y - x \)

\[
xy - y^2 = y - x
\]

\[
\frac{d}{dx} \left( xy \right) - \frac{d}{dx} \left( y^2 \right) = \frac{d}{dx} [y] - \frac{d}{dx} [x]
\]

\[
(u'v + uv') - 2y \frac{dy}{dx} = \frac{dy}{dx} - 1
\]

\[
\begin{align*}
  u(x) &= x \\
  u'(x) &= 1 \\
  v(x) &= y \\
  v'(x) &= \frac{dy}{dx}
\end{align*}
\]

\[
y + x \frac{dy}{dx} - 2y \frac{dy}{dx} = \frac{dy}{dx} - 1
\]

\[
\frac{dy}{dx} (x - 2y - 1) = -y - 1
\]

\[
\frac{dy}{dx} = \frac{-y - 1}{x - 2y - 1}
\]