\[ \frac{dV}{dt} = 2.5 \text{ ft}^3/\text{min} \]

Find \( \frac{dy}{dt} \) at \( y = 2 \)

\[ V = Bh = \frac{1}{2}bh \cdot 15 \]

\[ V = \frac{15}{2} xy \]

\[ \frac{4}{3} = \frac{x}{y} \quad V = \frac{15}{2} \cdot \frac{4}{3} y \cdot y \]

\[ \frac{4}{3} y = x \quad V = 10 y^2 \]

\[ \frac{dV}{dt} = 20y \frac{dy}{dt} \]

\[ 2.5 = 20 \cdot (2) \frac{dy}{dt} \quad \Rightarrow \quad \frac{dy}{dt} = \frac{2.5}{40} = 0.0625 \text{ ft/min} \]
Solve the problem.

1) Given the revenue and cost functions \( R = 26x - 0.5x^2 \) and \( C = 6x + 15 \), where \( x \) is the daily production, find the rate of change of profit with respect to time when 20 units are produced and the rate of change of production is 8 units per day.

\[
p = R - C = 26x - 0.5x^2 - (6x + 15)
\]

\[
p = 20x - 0.5x^2 - 15
\]

\[
\frac{dp}{dt} = 20 \frac{dx}{dt} - x \frac{dx}{dt}
\]

\[
\frac{dx}{dt} = 8 \quad x = 20
\]

\[
\frac{dp}{dt} = 20(8) - 20(8) = 0 = 0
\]
2) Water is being drained from a container which has the shape of an inverted right circular cone. The container has a radius of 4.00 inches at the top and a height of 5.00 inches. At the instant when the water in the container is 1.00 inches deep, the surface level is falling at a rate of 0.6 in./sec. Find the rate at which water is being drained from the container.

\[ V = \frac{1}{3} \pi r^2 h \Rightarrow V = \frac{1}{3} \pi x^2 y \]

\[ V = \frac{1}{3} \pi \left(\frac{4}{5} y\right)^2 y \Rightarrow V = \frac{16 \pi y^3}{75} \]

\[ \frac{dV}{dt} = \frac{48}{75} \pi y^2 \frac{dy}{dt} \]

\[ \frac{dV}{dt} = \frac{16}{25} \pi \left(\frac{4}{5} y\right)^2 (-0.6) = -\frac{48}{125} \pi \text{ in}^3/\text{sec} \]

\[ V = -0.384 \pi \text{ in}^3/\text{sec} \approx -1.206 \text{ in}^3/\text{sec} \]
3) The volume of a sphere is increasing at a rate of 8 cm$^3$/sec. Find the rate of change of its surface area when its volume is $\frac{32\pi}{3}$ cm$^3$. (Do not round your answer.)

We know $\frac{dV}{dt} = 8$ cm$^3$/sec. \[ V = \frac{32\pi}{3} \]

Find $\frac{dA}{dt}$

\[ A = 4\pi r^2 \]

\[ \frac{dA}{dt} = 8\pi r \frac{dr}{dt} \]

\[ \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \]

\[ 8 = 4\pi (2)^2 \frac{dr}{dt} \]

\[ \frac{1}{2\pi} = \frac{dr}{dt} \]

\[ = 4 \text{ cm}^2/\text{sec.} \]
Find an equation for the line tangent to given curve at the given value of \( x \).

1) \( y = \frac{x^2}{2}; \ x = 3 \)

\[ m = \frac{dy}{dx} = \frac{2x}{2} = x = 3 \quad \text{at} \quad \left(3, \frac{9}{2}\right) \]

A) \( y = 6x - 4.5 \)
B) \( y = 3x - 9 \)
C) \( y = 3x + 4.5 \)
D) \( y = 3x - 4.5 \)

\[ y - 4.5 = 3(x - 3) \Rightarrow y = 3x - 9 + 4.5 \Rightarrow y = 3x - 4.5 \]

Find all values of \( x \) (if any) where the tangent line to the graph of the function is horizontal.

2) \( y = 2 + 8x - x^2 \)

\[ \frac{dy}{dx} = 8 - 2x = 0 \]

A) \(-8\)
B) \(8\)
C) \(-4\)
D) \(4\)

Solve the problem.

3) The total cost to produce \( x \) handcrafted wagons is \( C(x) = 120 + 2x - x^2 + 7x^3 \). Find the marginal cost when \( x = 5 \).

A) 517
B) 637
C) 980
D) 860

4) A ball is thrown vertically upward from the ground at a velocity of 97 feet per second. Its distance from the ground after \( t \) seconds is given by \( s(t) = -16t^2 + 97t \). At what rate is the ball moving 4 seconds after being thrown?

A) 33 ft per sec
B) 132 ft per sec
C) \(-31\) ft per sec
D) \(-43\) ft per sec

Find the derivative.

5) \( y = \sqrt{4x + 2} \)

A) \( \frac{dy}{dx} = \frac{8}{2} \)
B) \( \frac{dy}{dx} = \frac{2}{2} \)
C) \( \frac{dy}{dx} = \frac{1}{2} \)
D) \( \frac{dy}{dx} = \frac{4}{2} \)
Find the derivative.

5) \( y = \sqrt{4x + 2} \)

A) \( \frac{dy}{dx} = \frac{8}{\sqrt{4x + 2}} \)

B) \( \frac{dy}{dx} = \frac{2}{\sqrt{4x + 2}} \)

C) \( \frac{dy}{dx} = \frac{1}{\sqrt{4x + 2}} \)

D) \( \frac{dy}{dx} = \frac{4}{\sqrt{4x + 2}} \)

\[ \frac{dy}{dx} = \frac{1}{2} (4x + 2)^{-\frac{1}{2}} \cdot 4 = \frac{2}{\sqrt{4x + 2}} \]
Sec. 3.2 Extrema (Extreme Values) of a Function

Global Extrema:
Let \( f \) be a function defined on an interval \( I \) containing \( c \).
1. \( f(c) \) is an absolute minimum of \( f \) on \( I \) if \( f(c) \leq f(x) \) for all \( x \) in \( I \).
2. \( f(c) \) is an absolute maximum of \( f \) on \( I \) if \( f(c) \geq f(x) \) for all \( x \) in \( I \).

Critical Number: An \( x \)-value where a maximum or a minimum can occur.
Critical numbers occur at endpoints and at points where the graph changes direction:
- Turning point
- Singularity point (a point where the function is continuous but has no derivative)

Note:
Maxima and minima are \( y \)-values.
The Extreme Value Theorem
If \( f \) is a continuous function on the closed interval \([a, b]\), then \( f \) must have both a (global) maximum and a (global) minimum.

Note: Two important conditions:
- Function must be continuous on the whole interval.
- The interval must be closed.

\[
\text{f is continuous on } [a, b] \\
\text{Minimum is } f(a) \\
\text{Maximum is } f(x_3)
\]

Local (Or Relative) Extrema
Let \( f \) be a function defined at \( x = c \).
1. \( f(c) \) is a relative maximum if there is an open interval \((a, b)\) containing \( c \) such that \( f(x) \leq f(c) \) for all \( x \) in \((a, b)\).
2. \( f(c) \) is a relative minimum if there is an open interval \((a, b)\) containing \( c \) such that \( f(x) \geq f(c) \) for all \( x \) in \((a, b)\).
The First Derivative Test:
1. Find all critical numbers: Endpoints, turning points (derivative = 0) and singularity points (derivative Does Not Exist).

2. Set up a table with a space before and after each internal critical number, and a space after the left endpoint and before the right endpoint (if any).

3. Find the derivative of a representative x-value between the critical numbers.

4. If the derivative is positive, then f is increasing in that interval (between the critical numbers). If the derivative is negative, then f is decreasing in that interval.

5. If the graph increases before and decreases after a critical number, then the critical number marks a local maximum. If the graph decreases before and increases after a critical number, then the critical number marks a local minimum.

Ex. 3.2 (p. 223): Find all relative extrema for the function.

\[ f(x) = -4x^2 + 4x + 1 \]

C.N.: \[ f'(x) = -8x + 4 \]
Set \[ f'(x) = 0 \] \[ \Rightarrow -8x + 4 = 0 \]
\[ x = \frac{1}{2} \]

So turning pt. \( x = \frac{1}{2} \)

Since \( f'(x) > 0 \) when \( x < \frac{1}{2} \) and \( f'(x) < 0 \) when \( x > \frac{1}{2} \),
then relative max. at \( x = \frac{1}{2} \), rel. max. is \( f(\frac{1}{2}) = 2 \)
Ex. 3.2 (p. 223): Find all relative extrema for the function.

8) \( h(x) = 2(x-3)^3 \)

C.N.: Turning pt.: \( h'(x) = 6(x-3)^2 \)

Set \( h'(x) = 0 \) \( \Rightarrow \frac{6(x-3)^2}{6} = 0 \) \( \Rightarrow \sqrt{(x-3)^2} = 0 \)

\( x - 3 = 0 \) \( \Rightarrow x = 3 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h'(x) )</td>
<td>+</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>

Since the \( h'(x) > 0 \) before and after \( x = 3 \), then the graph does not change directions. So no extremum.

\( h'(2) = 6(2-3)^2 = 6 \)

\( h'(4) = 6(4-3)^2 = 6 \)
Ex. 3.2 (p. 223): Find all relative extrema for the function.

\[ h(x) = \frac{4}{x^2 + 1} = 4(x^2 + 1)^{-1} \]

\[ h'(x) = -4(x^2 + 1)^{-2} \cdot 2x = \frac{-8x}{(x^2 + 1)^2} \]

Set \( h'(x) = 0 \):

\[ \frac{-8x}{(x^2 + 1)^2} = 0 \Rightarrow \frac{-8x}{(x^2 + 1)^2} \text{ top} \]

Is there any point where \( h'(x) \) DNE? \( \text{NO} \).

\[ (x^2 + 1)^2 = 0 \Rightarrow x^2 = -1 \text{ not real.} \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h(x) )</td>
<td>+</td>
<td>0</td>
<td>-</td>
</tr>
</tbody>
</table>

Relative maximum at \( x = 0 \) because \( h'(x) > 0 \) for \( x < 0 \) (increasing)

\[ h'(x) < 0 \text{ for } x > 0 \text{ (decreasing)} \]

Relative max is \( h(0) = 4 \).
Ex. 3.2: Find the absolute extrema of the function on the closed interval.

22) \( f(x) = x^2 + 2x - 4 \) on \([-1, 1]\)

Calculate the endpoints:
\[ x = -1, \quad x = 1 \]

Let's find the turning point(s):

\[ f'(x) = 2x + 2 \]

Set \( f'(x) = 0 \):
\[ 2x + 2 = 0 \quad \Rightarrow \quad x = -1 \]

Evaluate the function at the critical point and endpoints:

\[ f(-1) = (-1)^2 + 2(-1) - 4 = -5 \]

\[ f(1) = 1^2 + 2(1) - 4 = -1 \]

So, absolute maximum is \(-1\) at \( x = 1 \)

Absolute minimum is \(-5\) at \( x = -1 \)
Sec. 3.3: Concavity and the Second Derivative Test.

The Relationship among a function, its derivative and its second derivative:
- The second derivative is the derivative of the (first) derivative.
- The first derivative tells whether the function is increasing or decreasing:
  - If first derivative is negative, then function is decreasing
  - If first derivative is positive, then the function is increasing.

- The second derivative is to the first derivative what the first derivative is to the function.
- The second derivative tells whether or not the first derivative is increasing or not (the slope of the slopes).

**Case 1:** \( f'' < 0 \) **CONCAVE DOWN**
- Slopes are getting less and less (more and more negative)
- The graph is slowing down its rate of increase, or increasing its rate of decrease:

**Case 2:** \( f'' > 0 \) **CONCAVE UP**
- Slopes are getting greater (more and more positive)
- The graph is increasing its rate of increase, or slowing down its rate of decrease:

![Graph showing concave up and concave down cases with slopes and arrows indicating changes in behavior.]
Basis of the Second Derivative Test:
- When the second derivative is positive, the graph of $f$ is concave up.
- When the second derivative is negative, the graph of $f$ is concave down.
- What happens at turning points?

**Case 1: Local Minimum:**
- Concave up.
- First derivative is zero (turning point)
- Second derivative is positive (concave up)

**Case 2: Local Maximum:**
- Concave down.
- First derivative is zero (turning point)
- Second derivative is negative (concave down)

Note:
If both $f'$ and $f''$ are zero, then we cannot draw a conclusion from this approach.

**Inflection Point:** $f'' = 0$
A point where the graph changes concavity: from concave up to concave down, or from concave down to concave up.
Ex. 3.3 (P. 232): Find the point(s) of inflection of the graph of the function.

32) \( f(x) = x(6-x)^2 \)

\[ u = x \quad u' = 1 \]
\[ v = (6-x)^2 \quad v' = 2(6-x)(-1) = -2(6-x) = -12 + 2x \]

\[ f'(x) = u'v + uv' \]
\[ f'(x) = (6-x)^2 + x(-12+2x) = (6-x)^2 - 12x + 2x^2 \]

\[ f''(x) = 2(6-x)(-1) - 12 + 4x = -12 + 2x - 12 + 4x \]
\[ f''(x) = -24 + 6x \]

Set \( f''(x) = 0 \implies -24 + 6x = 0 \)
\[ x = 4 \]

\[ f'''(3) = -24 + 6(3) = -6 \quad \text{Concave down} \]
\[ f''(5) = -24 + 6(5) = 6 \quad \text{Concave up} \]

So Inflection Pt: \((4, f(4)) = (4, \underline{16})\)